MIXED RADIATION-CONVECTION BOUNDARY LAYER FLOW OF AN OPTICALLY DENSE FLUID ALONG A VERTICAL FLAT PLATE IN A NON-DARCY POROUS MEDIUM

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The combined effects of thermal radiation flux, thermal conductivity, Reynolds number and non-Darcian (Forcheimer drag and Brinkman boundary resistance) body forces on steady laminar boundary layer flow along a vertical surface in an idealized geological porous medium are investigated. The classical Rosseland one-dimensional diffusion approximation is implemented in the energy equation to avoid solving the general integro-differential equation for radiative transfer. Pseudo-similarity transformations are invoked and the resulting highly coupled and non-linear set of ordinary differential equations for momentum and energy equations are solved numerically using a well-tested and highly accurate shooting Runge-Kutta quadrature with a Merson-Gill algorithm. It is shown that the dimensionless velocity functions generally increase with rising radiation parameter and the Prandtl number, and the dimensionless temperature functions decrease as the non-Darcian body forces decrease. It is also shown that the dimensionless temperature functions rise in magnitude with rising radiation parameter and the Prandtl number but are depressed by lowered non-Darcian resistance parameter and rising Reynolds number. Generally radiation is seen to substantially boost the overall heat transfer.

Key words: porous medium, non-Darcy model, thermal radiation, mixed convection, numerical results.

1. Introduction

The analysis of radiation heat transfer effects on convective flows has been an active area of scientific endeavor for several decades. Original studies were inspired by technological developments in rocket propulsion systems, cosmic flight aerodynamics, spacecraft re-entry aerothermodynamics, plasma physics, glass production and furnace engineering. Major studies were subsequently performed over two decades ago by Cess (1966), Sparrow and Lin (1965), Tabaczynski and Kennedy (1967), Arpaci (1968), Taitel and Harnett (1968), Adunson and Gebhart (1972), England and Emery (1969), and Rhodes and Chen (1974). Later, flat plate coupled radiation-convection flows were examined by Bankston et al. (1977). Chang et al. (1983) studied the radiation-natural convection interactions in two-dimensional complex enclosures. A transient analysis of free convection-radiation transfer was carried out for double-infinite plates by Pop and

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Na (1984). Hirano et al. (1966) considered non-gray gas radiative channel flow, while Timofeyev (1991) investigated the radiative-convection flow on a thin plate and Wu et al. (1989) studied the radiative heat transfer in anisotropically scattering boundary layer flows with diffusely reflecting boundaries. Following the increase research in convective flows in porous media in the last four decades, considerable interest has emerged in the study of radiation transfer, radiation-conduction transfer and radiation convection transfer also in porous media. This has been largely due to the growing need to improve physical and mathematical models of radioactive subterranean repositories for civil engineers and hydrogeologists, thermal insulation systems in modern building complexes and geothermal energy systems in proximity to naturally occurring radioactive materials in the earth (see Gebhart et al., 1988) as well as novel developments in the electronics industry and scientific fields, such as cosmical hydrodynamics and astrophysics, as investigated by, for example, Sharma and Singh (1980). Numerous studies of porous radiative transport have been made following a very early analysis by Larkin and Churchill (1959). Chan and Tien (1974) studied radiative transfer in packed spheres. Kudo et al. (1985) performed a Monte Carlo analysis of radiation transport again in packed spherical porous media. Ozil and Birkebak (1977) analyzed the effect of environmental radiation on fibrous porous insulation. Yang et al. (1983) considered the radiative transfer in randomly packed spheres again with a Monte Carlo simulation.

Conversely, studies of buoyancy flows driven by convection and radiation in porous media are relatively rare in the literature. This is largely due to the inherent difficulty of tackling the non-linear convective flow terms in the boundary layer equations which are significantly more difficult to deal with than conduction-radiation or even purely radiative transfer problems which often yield exact analytical solutions. Recourse has to be made, therefore, to very powerful numerical techniques in addition to physical simplifications to yield solutions. The presence of a porous medium complicates issues by invoking extra body force terms such as inertial drag forces, viscous boundary friction and bulk matrix resistance which can considerably affect all components of heat transfer: conductive, convective and radiative. Several studies of convective-radiative boundary layer flows in porous media have been made, principally in last years. Chandra-ekhara and Nagaraju (1988) examined the convective-radiative boundary layer flow of an optically large density gray fluid past a horizontal flat plate embedded in a porous medium. Beg et al. (1996) investigated the convective-radiative flow of a dissipative gray gas in a non-Darcy porous medium with the Cogley-Vicenti-Giles differential near-equilibrium model. Hossain and Takhar (1996), and Hossain and Pop (2001) have studied the free and mixed convection-radiation interaction on boundary layer along a vertical flat plate embedded in a porous medium with high porosity. Radiation was, therefore, seen to substantially enhance the convective heat transfer in porous media.

In the majority of the porous convective-radiative transfer studies, the basic Darcy model for porous hydrodynamics has been applied. The present study is, therefore, motivated by the need to study convective-radiative flow in a non-Darcy porous medium and in this light, the Darcy-Brinkman-Forchheimer non-linear flow model has been employed. In addition, the fluid-porous medium system is modeled with a numerically variable thermal conductivity ratio since this property is significant in determining local heat transfer rates in the vicinity of nuclear repositories and in complex flows in thermo-hydrological systems, e.g. dikes and in melting and crystallizing volcanic chambers.

2. Physical regime and transport model

The geometry of the problem (Fig.1) comprises a heated vertical isothermal surface embedded in a saturated, homogenous and isotropically scattering porous medium and subjected to a radiation source, where x and y axes are measured along the plate and normal to it, respectively. Consider the steady mixed non-Darcy convective boundary layer flow of an optically dense fluid past the heated surface. The fluid density $\rho$ varies linearly with temperature according to the Boussinesq approximation, and the Grashof number $Gr$ is sufficiently large for the boundary layer approximation to be valid. The flow is laminar with free stream velocity $U_\infty$ and free stream temperature $T_\infty$. It is also assumed that the plate temperature $T_w$ is of sufficient high magnitude that the radiative heat transfer becomes significant. In practical geophysical flows this may be realized in a hyperthermal field with a stratum in proximity to super-heated magmatic chambers. The reader is referred to the paper by Huppert (1986) for geothermodynamical details. Local
thermodynamic equilibrium between the solid and fluid phases is assumed to hold in the analysis. We use a Darcy-Brinkman-Forchheimer model for the momentum equation as followed by Vafai and Tak (1981), and Hossain et al. (1994). It is worth mentioning that whilst Darcy's law is valid for lower Reynolds numbers \( \mathrm{Re} < 10 \), it fails to accommodate the Forchheimer drag effect at higher velocities and equally the Brinkman momentum diffusion caused by vorticity breakdown at the solid confining boundary, e.g., a stratum in geothermal system. Vafai (1984) has demonstrated that modeling the Darcy and Brinkman resistance as the same order of magnitude is an effective way of describing realistically the boundary effects. A whole range of mathematical techniques for simulating porous media effects are currently in use, such as hydraulic radius, homogenization, spatially periodic transport and random structure models. All these models are primarily focused on mimicking the geometrical characteristics of porous media. The present study has utilized, however, the drag force approach since this is most easily accommodated within the framework of classical boundary layer theory.

![Diagram](image)

**Fig.1. Physical model and coordinate system.**

To simplify the mathematics involved by the consideration of radiation heat transfer, which amounts to a non-linear integro-differential equation, we shall adopt the approach proposed originally for stellar physical problems by Rosseland (1936). This amounts to re-casting the energy equation in terms of a Fourier-type heat conduction expression. Implicit in this approximation is the necessary condition that the medium is optically dense so that radiation propagates a small distance prior to scattering or absorption. Major attenuation of the radiation takes place validating a diffusion model. The radiative heat flux which normally appears in the form of a three-dimensional energy source term \( \nabla q_r \) (designating the net thermal radiation energy acquired by a volume element of the fluid) can, therefore, be simplified to a one-dimensional radiative flux gradient term \( \partial q_{r,y} / \partial y \) in the \( y \)-direction. Siegel and Howell (1993) have shown that mathematically this implies that \( \partial q_{r,x} / \partial x \ll \rho C_p \mu \partial T / \partial x \), where \( \partial q_{r,x} / \partial x \) denotes the radiative flux gradient term in the \( x \)-direction. Effectively, the radiative diffusion adds a radiative conductivity to the
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ordinary fluid-saturated porous medium thermal conductivity. The simplicity of this model makes it particularly appropriate for the non-linear thermal flow problems such as the present one. The Rosseland (1936) model is preferable to the spherical harmonics model or the more complex discrete ordinate model, both of which transfer the general equation of radiative transfer (for a gray system on a spectral basis) into a set of simultaneous partial differential equations. These models are more accurate but very difficult to solve numerically since supplementary equations to the momentum and energy equations have to be solved simultaneously. A thorough review of these and other radiation models used in porous convection studies has recently been presented by Beg and Takhar (1997).

Having in view the above simplifications, the governing boundary layer equations for the present problem, can be shown to take the following form with respect to a Cartesian coordinate \((x, y)\) system (see Beg, 1996):

- continuity equation

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}
\]

- Darcy-Brinkman-Forchheimer momentum equation

\[
\frac{\rho}{\varepsilon^2} \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = -\frac{\mu}{K} u + \rho g \beta (T - T_\infty) + \frac{\mu^*}{\varepsilon} \frac{\partial^2 u}{\partial y^2} - \frac{\rho b}{K} u^2, \tag{2.2}
\]

- energy equation

\[
\left( \frac{\rho C_p}{\varepsilon} \right) \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \frac{16 \sigma T^4}{3a_r} + k_m \frac{\partial T}{\partial y} \right], \tag{2.3}
\]

where \(16 \sigma T^4/(3a_r)\) is the Rosseland diffusion term. All the quantities, which appear in these equations are defined in the nomenclature. The boundary conditions for the velocity field are

\[
y = 0: \quad u = v = 0, \tag{2.4}
\]

\[
y \to \infty: \quad u \to U_\infty, \tag{2.4}
\]

while for the thermal field the boundary conditions are

\[
y = 0: \quad T = T_\infty, \tag{2.5}
\]

\[
y \to \infty: \quad T \to T_\infty. \tag{2.5}
\]

We now proceed with the analysis of these equations and boundary conditions by introducing a pseudo-similarity transformation as proposed by Beg (1996)

\[
\xi = x/L, \quad \eta = y/\delta(x), \quad f(\xi, \eta) = \psi/(U_\infty \delta(x)), \quad 0(\xi, \eta) = (T - T_\infty)/(T_\infty - T_\infty) \tag{2.6}
\]

where \(L\) is a characteristic length of the plate and \(\delta(x) = (2\alpha(x - x_0)\nu/U_\infty)^{1/2}\) is the boundary layer thickness. Thus, Eqs. (2.1) to (2.4) become:

- momentum boundary layer
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\[ f'''' + \frac{\alpha}{\Lambda e^2} f f'' + \frac{2\alpha}{\Lambda D a Re} \xi f' - 2 \frac{\alpha F s}{\Lambda D a} \xi (f')^2 + 2 \frac{\alpha G r}{\Lambda e^2} \xi 0 = \frac{2\alpha}{\Lambda e^2} \xi \left( f, \frac{\partial^2 f}{\partial x^2}, f^2 \frac{\partial f}{\partial x} \right), \]  

(2.7)

- energy equation

\[ \left( \frac{1}{Pr K} \right) \left[ 1 + \frac{4}{3 N_R} \theta^2 \right] \theta'' + \alpha F s \theta'' + \left( \frac{4}{N_R} \right) \theta^2 \left( \theta'\right)^2 = 2 \alpha G r \left( f, \frac{\partial \theta}{\partial x}, \theta' \frac{\partial \theta}{\partial x} \right), \]  

(2.8)

while the boundary conditions Eqs.(2.4) and (2.5) read

\[ f(\xi,0) = 0, \quad f'(\xi,0) = 0, \quad \theta(\xi,0) = 1, \quad f'(\xi,\infty) = 1, \quad \theta(\xi,\infty) = 0 \]  

(2.9)

where \( \alpha \) is the boundary layer parameter, \( N_R = 4\alpha T_m^3 / 3 \), \( Ro = 4 / N_R \) is the Rosseland parameter, \( Da = K/\ell^2 \) is the Darcy number, \( F s = b/L \) is the Forchheimer number, \( Pr = \nu / \alpha_m \) is the Prandtl number, \( \Lambda = \mu^*/\mu \), \( \kappa = k_f / k_m \), \( Re = U_\infty L / \nu \) is the Reynolds number and \( Gr = g\beta (T - T_m) L^3 / \nu^2 \) is the Grashof number.

3. Solution method

Equations (2.7) and (2.8) with the boundary conditions Eq.(2.9) can be solved numerically using, for example, the Keller-box method. However, we shall use here a series expansion method for the reduced stream function \( f \) and the reduced temperature function \( \theta \) valid for small values of \( \xi (\ll 1) \) as follows

\[ f(\xi,\eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots, \]  

(3.1)

\[ \theta(\xi,\eta) = \theta_0(\eta) + \xi \theta_1(\eta) + \xi^2 \theta_2(\eta) + \ldots. \]  

Substituting these series into Eqs.(2.7) to (2.9), we obtain the following three sets of ordinary differential equations:

- zeroth order

\[ f_0'''' + A(\alpha f_0) f_0'' = 0, \]  

(3.2)

\[ \left( \hat{R} o / \hat{A} \right) \theta_0'' + Pr K \alpha f_0 \theta_0' + \hat{R} o \left( \theta_0^3, \theta_0^2 \right) = 0. \]  

- first order

\[ f_1'''' + A(\alpha f_0) f_1'' + 3A(\alpha f_0) f_1' - 2A(\alpha f_0) f_1' + 2A(\alpha f_0) f_1' - 2A(\alpha f_0) f_1' = 0, \]

\[ \left( \hat{R} o / \hat{A} \right) \theta_1'' + \left( \theta_0^3, \theta_0^2 \right) \theta_1'' + Pr K \alpha f_0 \theta_1' - \alpha f_1 \theta_0' + - 2Pr K f_0 \theta_1 + \hat{R} o \left( 2\theta_0 \theta_0' \theta_1' + 2\theta_0 \theta_1 \theta_0^2 \right) = 0. \]  

(3.3)
\[ f_1(0) = 0, \quad f_1'(0) = 0, \quad \theta_1(0) = 0, \]
\[ f_1' \to 0, \quad \theta_1 \to 0 \quad \text{as} \quad \eta \to \infty, \]
\text{cont.}(3.3)

- second order

\[ f_2'' + 5A f_2' f_1' + 3A f_1'' f_1 + A f_0 f_1 - 4A f_0' f_1' - 2A A f_1' f_1' = 0, \]
\[ (R_0/3)(\theta_0 + \theta_0') \theta_0 + (R_0/3)(2\theta_0' \theta_1) \theta_1' + (R_0/3)(\theta_0' \theta_1') + I \theta_2', \]
\[ + 2A f_0 f_1' + 2A f_0 f_1' + 2A f_0 f_1' - 2A \theta_0 f_1 + \theta_0 f_1' = 0, \]
\[ f_2(0) = 0, \quad f_2'(0) = 0, \quad \theta_2(0) = 0, \]
\[ f_2' \to 0, \quad \theta_2 \to 0 \quad \text{as} \quad \eta \to \infty, \]

where \[ A = l/(\Lambda \epsilon^2), \quad C = F_S/(\Lambda \Delta) , \quad D = G_r/(\Lambda Re^2), \quad E = l/(\Lambda Re Da). \]

The system of six ordinary differential Eqs. (3.2) to (3.4) has been solved numerically using the standard Runge-Kutta quadrature with a Merson-Gill algorithm as described by Beg (1996). This method has been also used, for example, by Soundalgekar and Takhar (1977) for the magnetohydrodynamic wedge flow, Takhar and Soundalgekar (1992) for radiation flows and Gorla et al. (1986) for micropolar convection flows. The quadrature is extremely complex and invokes up to 40 constituent subroutines encapsulated in the principal subroutine DO2HBF provided by the NAG library software. Effectively, the system of six ordinary differential equations is reduced to 15 first order coupled ordinary differential equations in terms of the pseudo-similarity variable \( \eta \) yielding solutions for \( f, f', f'', \theta \) and \( \theta' \) over the range \((x, x_0)\). The system is recast in the form

\[ dY_i/dx = F_i(X, Y_1, Y_2, \ldots, Y_N), \quad i = 1, 2, 3, \ldots, N \]

and the differentials \( F_i \) are computed using a technique which evaluates the derivatives at a general point \( X \). The value of the function \( Y \) is computed using the following marching algorithm

\[ Y(x + H) = Y(x) + [H/6] \left[ K_1 + 4K_2 + K_3 \right] \]

where \( K_i \) are specifically defined functions of \( X \) and the hierarchy of \( Y \) functions. The reader is referred to Beg (1996) for details. In the program, initially \( N \) boundary values of the variable \( Y_i \) have to be provided and the scheme rectifies these by a Newtonian iterative procedure. The equations are numerically integrated forward in the \( \eta(X) \) direction, i.e. forward shooting, commencing from the specified (known) and estimated (unknown) values of \( Y_i \) at \( X \), to eventually reach a matching point \( M \). In a similar fashion, the scheme then integrates backwards from a point \( X_i \) to the matching point \( M \), i.e. backward shooting. The discrepancy between the forward and backward shot values of \( Y_i \) and \( M \) would ideally be zero for an exact solution but is not so for the first several shots. The procedure reiterates in order to minimize this difference and eventually reduces it to approximately zero, by evaluating corrections to the guessed boundary values, subject to a present convergence criterion. The tests for convergence and the perturbation of the boundary conditions are executed in heterogeneous form. For example, if the error estimation \( Y \) is denoted \( \text{ERROR} \), then the program performs the following test.
\[ \text{ABS}(\text{ERROR}_i) < \text{ERROR}_i + (I + \text{ABS}Y_i). \]  

(3.7)

The algorithm makes the test absolute for \( Y_i \ll 1 \), and relative for \( Y_i \gg 1 \). Convergence, however, may never be attained in the case of faulty initial approximations. Additionally, the computational times for integration of the ordinary differential equations as well as numerical stability can be highly sensitive to the equality of the initial estimates of the unspecified boundary conditions, as well as the location of the matching point \( M \) and the terminal point representing the arbitrary infinity. Several important subroutines, which are worth mentioning, are FCN, which evaluates the derivatives \( Y_r, F_0SZF \) - performs linear algebraic computations and XO4AAF - registers explanatory error messages. The number of unknown parameters \( N_f \) (boundary values) must not exceed the number of differential equations \( N \). If \( N_f < N \), then \( (N - N_f) \) equations of the system are not invoked in the marching process and the integration process is controlled by these equations; they are independent of the other parameters and the solutions of the other \( N_f \) equations, and are always specified first in the program. The auxiliary routine DO2SAF executes the iteration using a Jacobian matrix whose \((i, j)\)-th element depends on the derivative of the solution \( Y_{_i} \) with respect to the \( j \)-th boundary parameter, denoted \( p(j) \). This matrix is calculated by a robust numerical differentiation technique, which requires \( N_f \) integration of the multiple first order differential system. The Runge-Kutta-Gill-Merson quadrature integrates optimally in the direction of decreasing solutions. This directionality is regulated by the routine DO2BAF. Convergence is usually attained within step-length of 0.1 or 0.2 for \( X \) avoids numerical diffusion and instabilities. During the present computations the values of \( \Lambda, \varepsilon, \text{Gr} \) and \( \alpha \) were fixed at preselected values of 3.0, 0.8, 100 and 0.01, respectively to maintain numerical stability. An infinity \( n = 8.0 \) was used and compilation times averaged around 300 seconds on a VICTOR 486 DX2 66 MHz computer.

![Graph of \( \theta_0(\eta) \) with respect to \( \eta \) for various values of the Rosseland radiation-conduction parameter \( \text{Ro} \) with \( \alpha = 0.01, \text{Pr} = 1.0, \kappa = 1.0, \Lambda = 3.0, \varepsilon = 0.7, \text{Da} = 0.01, \text{Gr} = 100, \text{Re} = 1.0, \text{Fs} = 0.055 \).](image)

4. Results and discussion

Plots are shown depicting variation of velocity functions \( f_0', f_1', f_2' \) and temperature functions \( \theta_0, \theta_1, \theta_2 \) with respect to the coordinate \( \eta \), for various values of the parameters which govern this problem. The variation of these functions with several physical parameters \( \text{Ro}(=4/N_P), \text{Pr}, \kappa, \text{Da}, \text{Fs}, \) and \( \text{Re} \) for fixed values of the parameters \( \Lambda, \varepsilon, \alpha \) and \( \text{Gr} \) has been considered in the present study.
Figure 2 shows the variation of the first order temperature function $\theta_0$ versus $\eta$ for various values of the radiation (Rosseland) parameters $Ro = 4/N_R$. The Prandtl number was specified at $Pr = 1.0$ (water at $C^\circ$), viscosity ratio parameter at $\Lambda = 3.0$ and porosity at $\varepsilon = 0.7$ to stimulate sedimentary geothermal systems. It is seen that as $Ro$ is increased from $Ro=0.0$, implying no radiation and pure conduction-convection heat transfer, a great rise in the temperature function $\theta_0$ is witnessed.

Figs.3a.b. Variation of $f_1(\eta)$ and $\theta_1(\eta)$ with the coordinate $\eta$ for various values of the Rosseland radiation-conduction parameter $Ro$ with $\alpha = 0.01$, $Pr = 1.0$, $\kappa = 1.0$, $\Lambda = 3.0$, $\varepsilon = 0.7$, $Da = 0.01$, $Gr = 100$, $Re = 100$, $Fs = 0.055$.

Figs.3c.d. Variation of $f_2(\eta)$ and $\theta_2(\eta)$ with the coordinate $\eta$ for various values of the Rosseland radiation-conduction parameter $Ro$ with $\alpha = 0.01$, $Pr = 1.0$, $\kappa = 1.0$, $\Lambda = 3.0$, $\varepsilon = 0.7$, $Da = 0.01$, $Gr = 100$, $Re = 1.0$, $Fs = 0.055$. 
Figures 3a, b and 4a, b show the variation of the profiles of the velocity functions $f'_1, f'_2$ and temperature functions $\theta_1, \theta_2$ versus $\eta$ with $Ro$. Increasing values of $Ro$ induces a substantial increase in $f'_1$ and $\theta_1$; however $f'_2$ is reduced in magnitude with increasing $Ro$. On the other hand $\theta_2$ is, as expected, increased in magnitude.

Fig.5. Variation of $\theta_0(\eta)$ with the coordinate $\eta$ for various values of the thermal conductivity parameter $\kappa$ with $\alpha = 0.01, Pr = 1.0, \kappa = 1.0, \Lambda = 3.0, \varepsilon = 0.8, Da = 0.01, Gr = 100, Re = 1.0, Fs = 0.055, Ro = 0.5$.

Figs.6a,b. Variation of $f'_1(\eta)$ and $\theta'_1(\eta)$ with the coordinate $\eta$ for various thermal conductivity parameter $\kappa$ with $\alpha = 0.01, Pr = 1.0, \kappa = 1.0, \Lambda = 3.0, \varepsilon = 0.8, Da = 0.01, Gr = 100, Re = 1.0, Fs = 0.055, Ro = 0.5$.

Figures 5-7a, b illustrate the variation of the velocity $f'_1, f'_2$ and temperature $\theta_0, \theta_1, \theta_2$ functions versus $\eta$ with thermal conductivity ratio $\kappa = k_f / k_m$. A rise in $\kappa$ induces a fall in the temperature function.
\( \theta \) as \( K \) increases from 0.5 to 4.0, \( \Lambda \) being fixed at 3.0 and \( \varepsilon \) at 0.8 (loosely packed-soils); \( R_o \) is maintained at 0.5 which represents a fairly strong radiation flux \( (N_{R} = 8.0) \). Over the same increasing range of \( K \) (0.5 to 4.0) the velocity function \( f_1' \) increases notably in magnitude, as does \( f_2' \), although in the first case velocities are generally negative and in the latter they are positive for a maximum value of \( K = 4.0 \). In both cases, the dimensionless first and second order temperature functions \( \theta_1, \theta_2 \) are greatly increased in magnitude as \( K \) increases from 0.5 through 1.0, 1.5, 2.0 to 4.0.

Figs.7a,b. Variation of \( f_1' (\eta) \) and \( \theta_2 (\eta) \) with the coordinate \( \eta \) for various thermal conductivity parameter \( \kappa \) with \( \alpha = 0.01, \ P_r = 1.0, \ \kappa = 1.0, \ \Lambda = 3.0, \ \varepsilon = 0.8, \ D_a = 0.01, \ G_r = 100, \ R_e = 1.0, \ F_s = 0.055, \ R_o = 0.5 \).

Figs.8a,b. Variation of \( f_1' (\eta) \) and \( \theta_2 (\eta) \) with the coordinate \( \eta \) for various values of the Darcy, \( D_a \), and Forchheimer. \( F_s \) numbers with \( \alpha = 0.01, \ P_r = 1.0, \ \kappa = 1.0, \ \Lambda = 2.0, \ \varepsilon = 0.9, \ G_r = 100, \ R_e = 1.0, \ R_o = 0.5 \).
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The variation of the first and second order velocity $f'_1, f'_2$ and temperature $\theta_1, \theta_2$ functions versus $\eta$ for various values of the Darcy and Forchheimer parameters $D_a$ and $F_s$ are illustrated in Figs. 9a, b and 9a, b. We notice that as $D_a$ is increased from a low value of $D_a = 0.002$ to a relatively high value of $D_a = 0.05$ the permeability $K$ of the porous medium increases according to $K = D_a L^2$. However, simultaneously the inertial drag force increases owing to a rise in the Forchheimer number (quadratic drag $= F_s f'^2/(L D_a)$) and this manifests overall as an increased resistance to flow leading to a reduction in the dimensionless velocity functions $f'_1$ and $f'_2$. Similarly, $\theta_1$ and $\theta_2$ are also depressed since the conduction mode of heat transfer is superceded by convection heat transfer (and radiation) as the permeability increases since solid particle volume decreases, which is the basic medium of conduction.

Finally, we mention that we have also determined the variation of the dimensionless velocity functions $f'_1, f'_2$ and dimensionless temperature functions $\theta_1, \theta_2$ versus $\eta$ for various values of the Reynolds number $Re$. However, due to space limitation, we will not present here the corresponding graphs of these functions. It was shown that both $f'_1$ and $f'_2$ are decreased as $Re$ increases since the buoyancy force is inversely proportional to the square of $Re$ (and directly proportional to $\theta$) for fixed $Gr, \alpha, \Lambda$ and this force is reduced to a far greater extent than the reduction in Darcy resistance, which is inversely proportional only to $Re$ for given values of the parameters $D_a, \Lambda$ (and directly proportional to $f'$). The buoyancy force, which drives the mixed convection flow is greatly inhibited as $Re$ increases especially for higher values of $Re$ compared with the lesser effect on the Darcy bulk matrix resistance which implies that the fluid velocity decreases in magnitude. As expected, since $\theta$ is contained in the buoyancy force term, both $\theta_1$ and $\theta_2$ decrease in magnitude as $Re$ increases from 1.0 to 100.0.

5. Conclusions

1) Dimensionless velocity functions are generally increased by increasing the values of the parameters $Ro, Pr$ and $\kappa$. These profiles are reduced by increasing $D_a, F_s$ and $Re$. 
2) Dimensionless temperature functions are basically enhanced in magnitude with increasing the values of the parameters Ro, Pr and κ. These are depressed by rising Da, Fs and Re.

3) In particular, the radiation parameter Ro augments the overall heat transfer process and acts as an important supplementary mode to the conduction and convection in the porous medium.

Nomenclature

\( a_r \) – Rosseland reflection coefficient
\( b \) – Forchheimer constant
\( C_p \) – specific heat at constant pressure
\( Da \) – Darcy number
\( f \) – dimensional stream function
\( Fs \) – Forchheimer number
\( g \) – acceleration due to gravity
\( Gr \) – Grashof number
\( k_f \) – fluid thermal conductivity
\( k_m \) – thermal conductivity of the fluid-saturated porous medium
\( K \) – permeability of the porous medium
\( L \) – reference length
\( N_R \) – radiation parameter
\( Pr \) – Prandtl number
\( Re \) – Reynolds number
\( Ro \) – Rosseland parameter
\( T \) – temperature
\( T_w \) – wall temperature
\( T_\infty \) – free stream temperature
\( u, v \) – velocities components along \( x, y \) directions
\( U_\infty \) – free stream velocity
\( x, y \) – Cartesian coordinates along the plate and normal to it, respectively
\( \nu \) – fluid kinematic viscosity
\( \alpha \) – boundary layer parameter
\( \alpha_m \) – effective thermal diffusivity of the fluid-porous medium
\( \beta \) – expansion coefficient
\( \varepsilon \) – porosity of the porous medium
\( \Theta \) – dimensionless temperature
\( \kappa \) – thermal conductivity ratio
\( \Lambda \) – ratio of dynamic viscosities
\( \mu \) – fluid dynamic viscosity
\( \mu^* \) – modified viscosity
\( \delta \) – boundary layer thickness
\( \sigma \) – Stefan-Boltzmann constant
\( \xi, \eta \) – pseudo-similarity coordinates
\( \rho \) – density
\( \psi \) – stream function

Superscript

\( \cdot \) – differentiation with respect to \( \eta \)

Subscripts

\( w \) – wall condition
\( \infty \) – ambient condition
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References


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