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Three-Dimensional Micropolar Flow due to a Stretching Flat Surface†

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A numerical solution of the steady boundary layer equations under similarity assumptions is obtained for the three-dimensional flow of a micropolar fluid over a continuous stretching surface. The case when microrotation vector is zero on the solid surface is considered. Using properly similarity variables, the three-dimensional Navier – Stokes equations are reduced to a set of four coupled non-linear ordinary differential equations. A very efficient numerical solution has been used to solve the boundary layer equations and a comparison is made with earlier results for a Newtonian fluid.

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Introduction

The inadequacy of the classical Navier – Stokes theory to describe rheologically complex fluids such as liquid crystals, polymeric fluids, animal blood, concentration suspensions, porous media, lubrication, turbulent shear flows, etc., has led to the development of microcontinuum fluid mechanics as an extension of the classical theory. Many models have been proposed to take into account the mechanically significant microstructure of such fluids. Papers published before 1974 can be found in the review papers by Ariman et al. [1, 2] and after 1974 in the most recently published books by Łukaszewicz [3] and Eringen [4]. One of the models, the theory of micropolar fluids, introduced by Eringen [5, 6], has generated a lot of interest in literature. In this theory, the micropolar fluid exhibits the microrotational effects and microrotational inertia. Such fluids can support couple stress and body couples only. Physically they may represent fluids with bar-like or sphere-like elements, see Unsworth and Chiam [7]. The concept of boundary layer approximation in micropolar fluids past surfaces was introduced by Willson [8]. He obtained the appropriate two-dimensional boundary layer equations using an order-of-magnitude argument and neglecting

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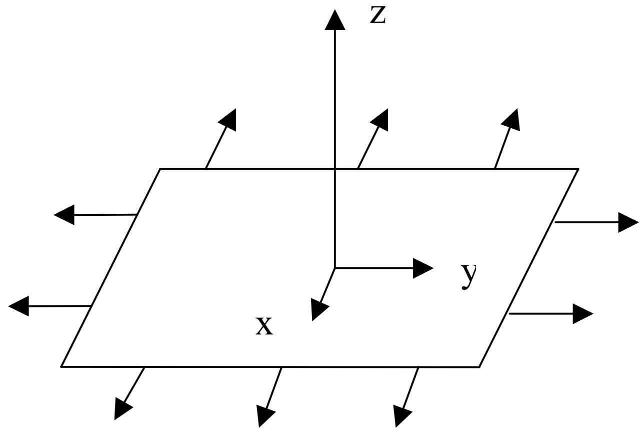

Fig. 1. Physical model and coordinate system.

certain microinertia terms. Peddiesen and McNitt [9], Guram and Smith [10], Gorla [11] and Lock et al. [12] applied the boundary layer theory to problems of steady-state flow of micropolar fluids near the two-dimensional stagnation points. Unsteady incompressible boundary layer flow of micropolar fluids at stagnation points was studied by and Kumari and Nath [13] and recently by Lock et al. [14, 15].

The fluid dynamics due to a stretching surface is important in extrusion processes. The production of sheeting material arises in a number of industrial manufacturing processes and includes both metal and polymer sheets. In the manufacture of the latter, the material is in a molten phase when thrust through an extrusion die and then cools and solidifies some distance away from the die before arriving at the collection stage. The region between the die and the collecting mechanism the material, while cooling, is found to stretch: because of the solidification that eventually occurs we may reasonably expect that the stretching process vary with distance from the die. It appears that Wang [16] was the first to present an exact similarity solution for the steady three-dimensional flow of a viscous and incompressible fluid due to a stretching flat surface. Then, Lakshmisha et al. [17] have considered the unsteady three-dimensional fluid motion caused by the time-dependent stretching of a flat permeable surface in the presence of a uniform magnetic field. The aim of this paper is to study the steady three-dimensional flow of a micropolar fluid due to a stretching flat surface. The equations of motion are reduced to a system of coupled nonlinear ordinary differential equations, which in turn is solved numerically for different combination of the material parameter K and the stretching ratios λ. The results are given for the values of λ in the range $0 \leq \lambda \leq 1$ and they are given both in tabular and graphical form. The accuracy of these results were checked by comparing them with the previous results for a Newtonian fluid $(K = 0)$. The comparison is excellent.

1. Basic Equations

We consider the laminar motion of a viscous and incompressible micropolar fluid caused by the stretching of an infinite flat surface in two lateral directions x and y as shown in Fig. 1. The surface is assumed to be highly elastic and is stretched by the action of uniform but increasing forces in the same or in opposite directions. The fluid is assumed to have constant properties, is at rest at infinity and the surface is assumed to be impermeable. With the usual boundary layer approximations the governing equations for the situation described here are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=(\mu+\kappa)\frac{\partial^2 u}{\partial z^2}-\kappa\frac{\partial h_1}{\partial z}\,, \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=(\mu+\kappa)\frac{\partial^2 v}{\partial z^2}+\kappa\frac{\partial h_2}{\partial z}\,, \tag{3}$$

$$\rho j\left(u\frac{\partial h_1}{\partial x}+w\frac{\partial h_1}{\partial z}\right)=\gamma\frac{\partial^2 h_1}{\partial z^2}-\kappa\left(2h_1+\frac{\partial v}{\partial z}\right) \tag{4}$$

$$\rho j\left(u\frac{\partial h_2}{\partial x}+w\frac{\partial h_2}{\partial z}\right)=\gamma\frac{\partial^2 h_2}{\partial z^2}-\kappa\left(2h_2-\frac{\partial u}{\partial z}\right) \tag{5}$$

where (x, y, z) are the Cartesian coordinates; (u, v, w) are the velocity components along (x, y, z)-axes; $(h1, h2, 0)$ are the microrotation components along (x, y, z)-axes; j is the microinertia density and r, μ, κ and γ are the density, dynamic viscosity, vortex viscosity and spin gradient viscosity, respectively. Here, the spin gradient viscosity γ is assumed to be a constant and given by, see Rees and Bassom [18] or Rees and Pop [19],

$$\gamma=(\mu+\kappa/2)j=\mu(1+K/2)j, \tag{6}$$

where $K=\kappa/\mu$ is the material parameter and we take $j=\nu/a$ as the length scale. Relation (6) is invoked to allow the field equations to predict the correct behavior in the limiting case when microstructure effects become negligible, and the microrotation components h_1 and h_2 reduce to the angular velocities.

We assume that the boundary conditions associated to Eqs (1) – (5) are given by

$$\begin{aligned}&u=ax, \quad v=by, \quad w=0, \quad h_1=h_2=0 \quad \text{on} \quad z=0,\\ &u=0, \quad v=0, \quad w=0, \quad h_1=h_2=0 \quad \text{as} \quad z\to\infty,\end{aligned} \tag{7}$$

where the constants a and b represent the stretching rate, in x- and y-directions having dimensions $(\mathrm{time})^{-1}$. It should be noticed that there is no general agreement as to what type of boundary condition one should use for the microrotation, and a number of plausible alternatives are possible, as discussed by Guram and Smith [10] or Łukaszewicz [3]. However, a common choice is to set $h_1(0)=h_2(0)=0$ called strong interaction by Guram and Smith [10].

We now introduce the following similarity transformation

$$\begin{aligned}&u=axf'(\eta), \quad v=byg'(\eta), \quad w=-(a\nu)^{1/2}(f+g),\\ &h_1=a(a/\nu)^{1/2}yF(\eta), \quad h_2=a(a/\nu)^{1/2}xG(\eta),\end{aligned} \tag{8}$$

where ν is the kinematic viscosity; η is the similarity variable, which is defined as $\eta=(a/\nu)^{1/2}z$ and primes denote differentiation with respect to η. Substituting Eq (8) into Eqs (1) – (5), we obtain the following set of nonlinear, coupled ordinary differential equations

$$(1+K)f'''-KG'=f'^2-(f+g)f'', \tag{9}$$

$$(1+K)g''' + KF' = g'^2 - (f+g)g'', \tag{10}$$

$$\left(1+\frac{K}{2}\right)F'' - K(g''+2F) = g'F - (f+g)F', \tag{11}$$

$$\left(1+\frac{K}{2}\right)G'' + K(f''-2G) = f'G - (f+g)G', \tag{12}$$

and the boundary conditions (7) become

$$\begin{gathered} f(0)=0, \quad g(0)=0, \quad f'(0)=1, \quad g'(0)=\frac{b}{a}=\lambda, \quad F(0)=0, \quad G(0)=0, \\ f'(\infty)=0, \qquad g'(\infty)=0, \qquad F(\infty)=0, \qquad G(\infty)=0, \end{gathered} \tag{13}$$

where λ is the stretching rate parameter and can have values between -1 and $+1$. When $\lambda = 1$, the problem is axisymmetric ($f = g, F = G$) and when $\lambda = 0$, we have the case of a two-dimensional flow ($g = G = 0$). We notice that the stretching rate parameter λ can take negative values when the nature of the forces in the x- and y-directions are opposite to each other. In this situation ($\lambda < 0$)), we have a saddle point flow, while for $\lambda > 0$ it is a nodal point flow (see Davey [20]). When $\lambda > 14$, the x- and y-axes can be interchanged. Further, we notice that for $K = 0$ (Newtonian fluid), the system of equations (9) – (12) becomes uncoupled and Eqs (9) and (10) reduce to those found by Wang [16].

Expressions for the wall skin frictions τ_{zx} and τ_{zy}, and also spin gradients at the wall m_{zx} and m_{zy} in the x- and y-directions are given by, see Guram and Smith [10],

$$\begin{aligned} \tau_{zx} &= -(\mu+\kappa)\left(\frac{\partial u}{\partial z}\right)_{z=0} = -(\mu+\kappa)ax\left(\frac{a}{\nu}\right)^{1/2} f''(0), \\ \tau_{zy} &= -(\mu+\kappa)\left(\frac{\partial v}{\partial z}\right)_{z=0} = -(\mu+\kappa)ay\left(\frac{a}{\nu}\right)^{1/2} g''(0) \end{aligned} \tag{14}$$

and

$$\begin{aligned} m_{zx} &= (\mu+\kappa)\left(\frac{\partial h_1}{\partial z}\right)_{z=0} = (\mu+\kappa)\left(\frac{a^2}{\nu}\right) yF'(0), \\ m_{zy} &= (\mu+\kappa)\left(\frac{\partial h_2}{\partial z}\right)_{z=0} = (\mu+\kappa)\left(\frac{a^2}{\nu}\right) xG'(0). \end{aligned} \tag{15}$$

2. Results and Discussion

Eqs (9) – (12) together with the boundary conditions (13) form a nonlinear two-point boundary value problem, which has been solved numerically using a very efficient method proposed by Blottner [21]. As in the paper by Wang [16] only the positive values (0 to 1) have been considered here for λ, and in this range solutions were obtained without any difficulty. The implicit finite-difference method discussed by Blottner [21] has proven to be accurate and adequate for the solution of coupled differential equations similar to Eqs (9) – (12). For this reason, it is employed in the present work. The third-order differential equations (9) and (10) were converted to second-order differential equations by a simple change of variables such as $V = f'$ and $H = g'$. The resulting equations in terms of V and H along with Eqs (11) and (12) are linearized and then discretized using three-point central difference quotients with variable step sizes in the η direction. The resulting equations form

Table 1
Comparison of $f''(0), g''(0)$, $f($ [16]) and $g($ [16]) for $K = 0$ (Newtonian fluid) with those by Wang [16].

λ	$f''(0)$	$g''(0)$	$f($ [16])	$g($ [16])
0.00	−1.000000	0.000000	1.000000	0.000000
0.25	−1.049355	−0.194742	0.906390	0.257730
	(−1.048813)	(−0.465205)	(0.907075)	(0.257986)
0.50	−1.093197	−0.465356	0.841686	0.451256
	(−1.093097)	(−0.465205)	(0.842360)	(0.451671)
0.75	−1.134800	−0.794579	0.791620	0.611586
	(−1.134485)	(−0.794622)	(0.792308)	(0.612049)
1.00	−1.174076	−1.174012	0.750818	0.750819
	(−1.173720)	(−1.173720)	(0.751527)	(0.751527)

Table 2
Effects of K and λ on $f''(0), g''(0)$, $f($ [16]), $g($ [16]), $F'(0)$ and $G'(0)$.

K	λ	$f''(0)$	$g''(0)$	$f($ [16])	$g($ [16])	$F'(0)$	$G'(0)$
0.0	0.00	−1.000000	0.000000	1.000000	0.000000	0.000000	0.000000
0.5	0.00	−0.805393	0.000000	1.182613	0.000000	0.000000	−0.175682
1.0	0.00	−0.684103	0.000000	1.316788	0.000000	0.000000	−0.242028
2.0	0.00	−0.540267	0.000000	1.532326	0.000000	0.000000	−0.285855
3.0	0.00	−0.456974	0.000000	1.711810	0.000000	0.000000	−0.292790
0.0	0.25	−1.049355	−0.194742	0.906390	0.257730	0.000000	0.000000
0.5	0.25	−0.845364	−0.156261	1.076752	0.304221	0.045333	−0.184006
1.0	0.25	−0.719046	−0.132406	1.201409	0.337686	0.060401	−0.254923
2.0	0.25	−0.568154	−0.104266	1.400074	0.391451	0.068726	−0.302272
3.0	0.25	−0.481098	−0.087891	1.564464	0.436460	0.068851	−0.310069
0.0	0.50	−1.093197	−0.465356	0.841686	0.451256	0.000000	0.000000
0.5	0.50	−0.882104	−0.374686	1.002237	0.535656	0.095162	−0.190711
1.0	0.50	−0.750476	−0.318145	1.119802	0.596802	0.129910	−0.265698
2.0	0.50	−0.594296	−0.251111	1.306283	0.694113	0.151303	−0.316438
3.0	0.50	−0.502603	−0.212036	1.459918	0.774676	0.153463	−0.325182
0.0	0.75	−1.134800	−0.794579	0.791620	0.611586	0.000000	0.000000
0.5	0.75	−0.916330	−0.641343	0.944369	0.728686	0.147435	−0.196373
1.0	0.75	−0.780461	−0.545591	1.056331	0.814126	0.204804	−0.275061
2.0	0.75	−0.617857	−0.431813	1.233351	0.949233	0.242717	−0.329052
3.0	0.75	−0.523272	−0.365302	1.378568	1.060371	0.248520	−0.338819
0.0	1.00	−1.174076	−1.174012	0.750818	0.750819	0.000000	0.000000
0.5	1.00	−0.948254	−0.948254	0.897083	0.897076	0.201270	−0.201270
1.0	1.00	−0.808342	−0.808342	1.004463	1.004454	0.283377	−0.201270
2.0	1.00	−0.641093	−0.641093	1.173586	1.173608	0.340510	−0.340509
3.0	1.00	−0.543074	−0.543074	1.312011	1.311990	0.351339	−0.351339

a tri-diagonal system of algebraic equations that can be solved by the well known Thomas algorithm (see Blottner [21]). The first-order differential equations $V = f'$ and $H = g'$ are discretized using the trapezoidal rule. Due to the nonlinearities of the equations, an iterative solution is required. For convergence, the maximum absolute error between two successive iterations was taken to be 10^{-6}. A starting step size of 0.001 in the η-direction with an increase of 1.04 times the previous step size was found to give accurate results. The total number of points in the η-direction was taken to be 196 to ensure proper approach of the solution to the free stream conditions. With these values, the value of h was approximated by the value $\eta = 52$. The accuracy of the aforementioned numerical method was validated by direct comparison with the numerical results reported earlier by Wang [16] for the case of Newtonian fluid $(K = 0)$. Table 1 presents the results of this comparison. It can be seen from the table that excellent agreement between the results exists. This favorable comparison lends confidence in the numerical results to be reported in the next section.

Numerical results are presented for the reduced skin frictions coefficients, $f''(0)$ and $g''(0)$, and also for the reduced spin gradients at the wall, $F'(0)$ and $G'(0)$, in the x- and y-directions in Table 2 for some values of the material parameter K and the stretching ratios λ. Results are also shown graphically for the velocity and microrotation profiles in Figs 2 to 7. Table 2 clearly shows that the quantities $f''(0)$, $g''(0)$, $F'(0)$ and $G'(0)$ are markedly affected by the parameter K in the micropolar model. The values of $f''(0)$ and $g''(0)$ for micropolar fluids decrease with the increase of K. This is because an increase in K would result in an increase in the total viscosity of the fluid flow, thus decreasing the skin friction coefficients. The values of $F'(0)$ and $G'(0)$ increase as the parameter K increases with $F'(0) > 0$, while $G'(0) < 0$ for the values of λ considered. Further, we can see from Figs 2 to 7 that the reduced velocity profiles f' and the reduced stream function f in the x-direction increase with the increase of K and the boundary layer thickness decreases as λ increases (Figs 2 and 7). It is also seen that the reduced velocity profiles g' and the reduced stream function g in the y-direction increases with K and also the boundary layer thickness increases with the increase of λ (Figs 3 and 5). On the other hand, we noticed that the reduced microrotation profiles F increase with both K and λ as can be seen from Fig. 6. These profiles reach a maximum at a finite distance from the wall and the boundary layer thickness increase as the maximum moves away from the wall. Finally, Fig. 7 shows that the reduced microrotation profiles G in the y-direction is negative and increases in absolute value with K but the boundary layer thickness decreases with the increase of K and λ.

Conclusions

In this study, the three-dimensional boundary layer flow of a micropolar fluid due to a stretching flat surface in two lateral directions has been considered. It is shown that the solutions are exact similarity solutions of the Navier – Stokes and microrotation equations. It is found that the numerical solution to the problem, augmented with a physically acceptable boundary conditions, gives a set of universal curves. These profiles show that the boundary layer structure is markedly affected by the material parameter K.

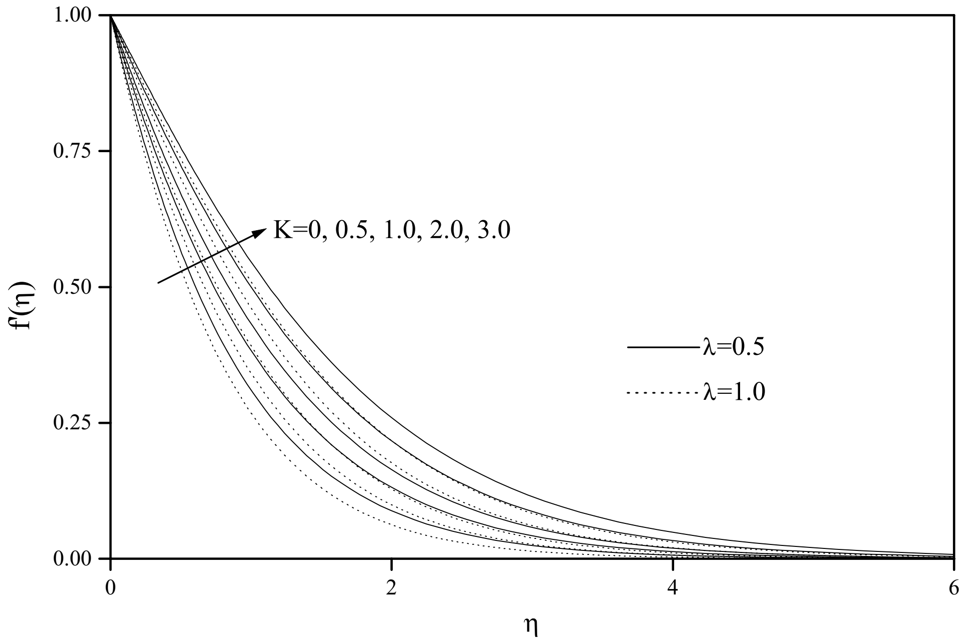

Fig. 2. Effects of K and λ on $f'(\eta)$.

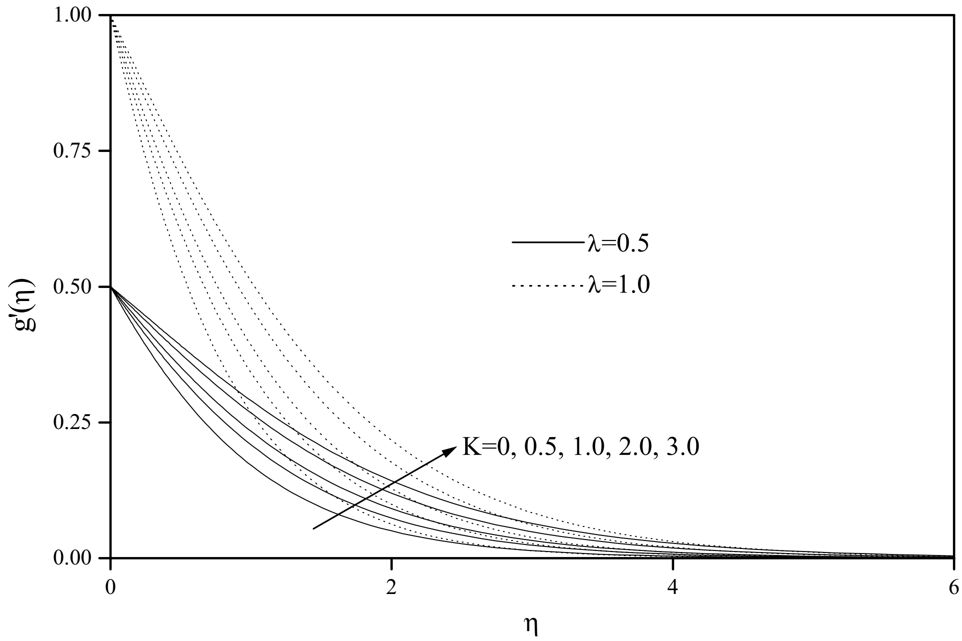

Fig. 3. Effects of K and λ on $g'(\eta)$.

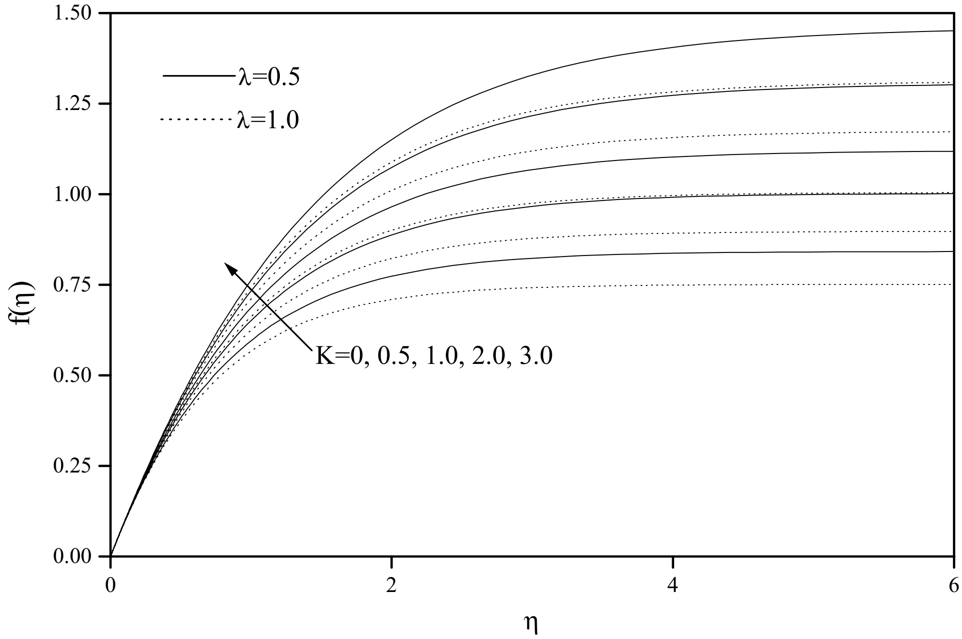

Fig. 4. Effects of K and λ on $f(\eta)$.

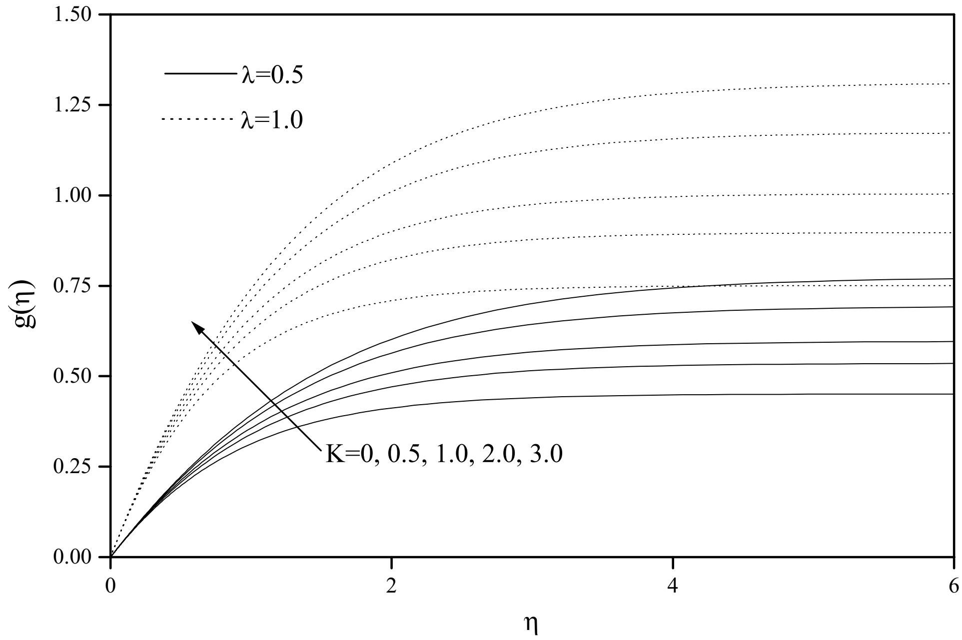

Fig. 5. Effects of K and λ on $g(\eta)$.

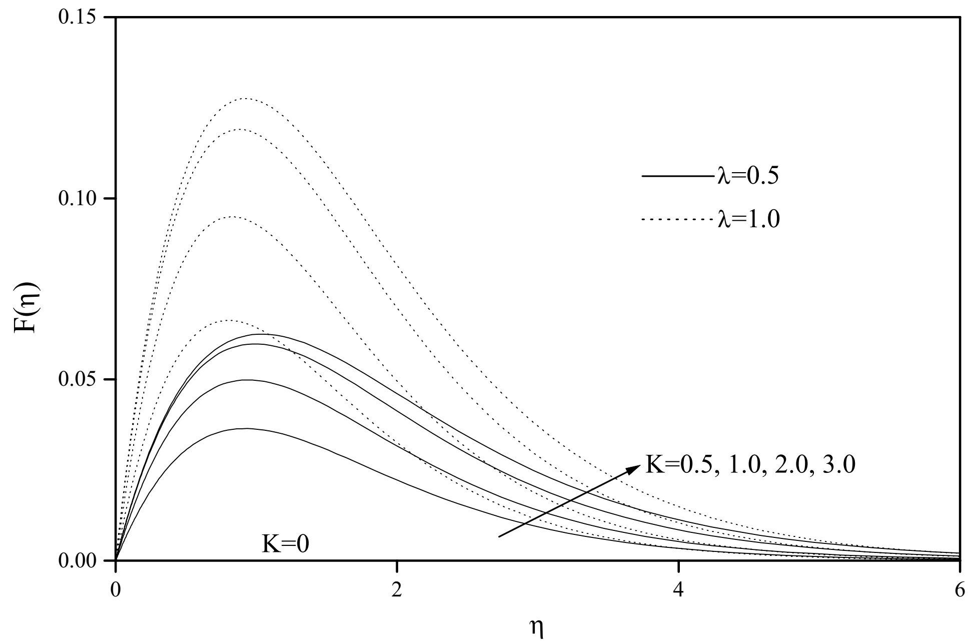

Fig. 6. Effects of K and λ on $F(\eta)$.

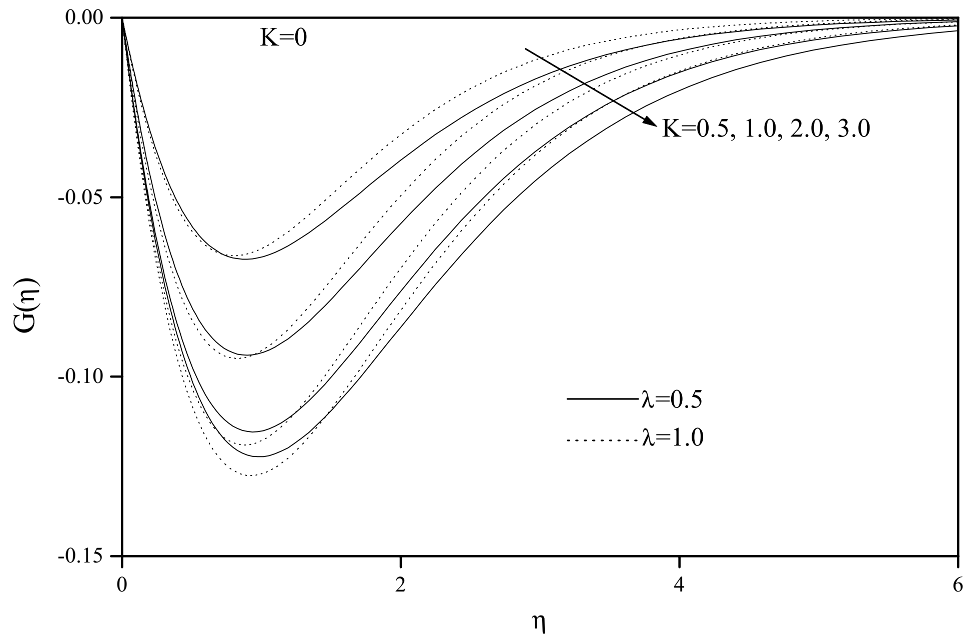

Fig. 7. Effects of K and λ on $G(\eta)$.

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