FREE CONVECTION FROM A VERTICAL CYLINDER EMBEDDED IN A POROUS MEDIUM FILLED WITH COLD WATER

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A free convection boundary layer flow along a heated vertical cylinder embedded in a porous medium saturated with pure or saline water at low temperatures, up to 20°C, is considered. The boundary layer analysis is formulated in terms of Darcy's law and a new density equation of state, which is of very high accuracy and of simple form, is postulated. Numerical solutions are presented and the flow field characteristics are analysed in detail for both cases of downward and upward flows. A very good agreement between the present results and those reported for particular situations was found.

Key words: free convection, vertical cylinder, porous medium, pure or saline water at low temperatures.

1. Introduction

Buoyancy driven convection in fluid saturated porous media originated from the empirical theory of Darcy and is supplemented by a suitable equation governing the transport of thermal energy. It has been an important area of research during the last several decades. This is due to an increasing interest in transport processes in a large number of geophysical and engineering applications, such as in geothermal reservoirs, petroleum extraction, fibre and granular insulation materials, storage of radioactive nuclear waste materials, packed-bed chemical reactors, irrigation systems, transpiration cooling, to name just a few applications. Some of the most important analytical, numerical and experimental studies with such applications, which present the current state-of-the-art in the area of convective heat transfer in porous media, have been gathered in the monographs by Nield and Bejan (1999), Ingham and Pop (1998; 2002), Yafai (2000) and Pop and Ingham (2001).

Studies of convective heat transfer in porous media have been carried out in the past using the Boussinesq approximation, namely the fluid density $\rho$ varies linearly with temperature. However, this is inappropriate for water at low temperatures because of the extremum at about 4°C in pure water at 1 atm. Such conditions occur commonly in porous media, such as permeable soils flooded by a cold lake or sea water, water-ice slurries, etc. A limited number of studies have been devoted to the problem of convective boundary layer adjacent to heated or cooled bodies immersed in a porous medium saturated with cold water wherein a density extremum may arise. It should be mentioned that the buoyancy flow with an extremum may become very complicated, with local flow reversals and convective inversions. Density differences may then not be expressed as a linear function of the temperature. Ramilison and Gebhart (1980) examined the possible similarity solutions for vertical, buoyancy induced flow in a porous medium saturated with cold water. The corresponding case of a horizontal surface has been considered by Lin and Gebhart (1986). Gebhart et al. (1983) obtained multiple steady state solutions for the problem considered by Lin and Gebhart.
(1986) using two numerical codes. A review of the convective flow in the vicinity of the maximum-density condition in water at low temperatures, along with relevant citations, is available in the survey by Kukulka et al. (1987).

The present paper deals with the free convection boundary layer adjacent to a heated or cooled vertical cylinder embedded in an extensive porous medium saturated with either pure or saline water under the conditions in which a density extremum might occur. The density state equation used here is that proposed by Gebhart and Mollendorf (1977), which has been shown to be very accurate for both pure and saline water to a pressure level of 1000 bars up to 20°C, and to 40% salinity. To the best of our knowledge, this problem has not been considered before. However, Minkowycz and Cheng (1976) made an analysis for free convection boundary layer about a vertical cylinder in a porous medium saturated with a Boussinesq fluid, where the surface temperature of the cylinder varies as \( x^\lambda \), a power function of the distance from the leading edge.

2. Basic equations

Consider a vertical cylinder of radius \( r_0 \) and at a constant surface temperature \( T_w \), which is embedded in a fluid-saturated porous medium of ambient temperature \( T_\infty < T_w \) saturated with salt water. The governing boundary layer equations are given by

\[
\begin{align*}
\frac{\partial}{\partial \bar{x}}(\bar{u} \bar{u}) + \frac{\partial}{\partial \bar{r}}(\bar{r} \bar{v}) &= 0, \quad (2.1) \\
\bar{u} &= \pm \frac{gK}{\mu} (\rho_\infty - \rho), \quad (2.2) \\
\frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{r}} &= \frac{\alpha_m}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial T}{\partial \bar{r}} \right) \quad (2.3)
\end{align*}
\]

where \( \bar{x} \) and \( \bar{r} \) are the axial and radial Cartesian coordinates, respectively, \( \bar{u} \) and \( \bar{v} \) are the velocity components in the \( \bar{x} \) and \( \bar{r} \) directions, respectively, \( T \) is the fluid temperature, \( K \) is the permeability of the porous medium, \( g \) is the magnitude of the gravitational acceleration, \( \rho, \mu \) and \( \alpha_m \) are the density, viscosity and effective thermal diffusivity of the porous medium. The plus sign in Eq.(2.2) is for the upward flow while the minus sign is for the downward flow, respectively. The new density equation which applies to both pure and saline water is given by

\[
\rho = \rho_m(s, p) \left[ 1 - \beta_m(s, p) \right] \left[ T - T_m(s, p) \right]^{\frac{1}{\gamma}} \quad (2.4)
\]

where \( \rho_m \) and \( T_m \) denote the maximum density and temperature, respectively, for given pressure and salinity levels. The forms and values of \( \gamma \), \( \beta_m \), \( \rho_m \) and \( T_m \) are given in the paper by Gebhart and Mollendorf (1977). Using relation (2.4), Eq.(2.2) can be written as follows

\[
\bar{u} = \pm \frac{\rho_m g K \beta_m}{\mu} \left[ T - T_m \right]^{\frac{1}{\gamma}} \left[ T_\infty - T_m \right]^{\frac{1}{\gamma}} \quad (2.5)
\]

Equations (2.1), (2.3) and (2.5) have to be solved subject to the boundary conditions.
\begin{equation}
\bar{v} = 0, \quad T = T_w \quad \text{on} \quad \bar{r} = r_0, \quad \bar{x} \geq 0,
\end{equation}
\begin{equation}
\bar{u} \to 0, \quad T \to T_w \quad \text{as} \quad \bar{r} \to \infty, \quad \bar{x} \geq 0.
\end{equation}

We introduce now the following non-dimensional variables
\begin{equation}
x = \bar{x}/r_0, \quad r = Ra^{1/2}(\bar{r}/r_0), \quad u = \bar{u}/U_c, \quad v = Ra^{1/2}(\bar{v}/U_c),
\end{equation}
\begin{equation}
\theta = (T - T_w)/(T_r - T_w), \quad R = (T_m - T_w)/(T_r - T_w)
\end{equation}
where \( U_c \) and \( Ra \) are the characteristic velocity and the Rayleigh number, respectively, which are defined as follows
\begin{equation}
U_c = \frac{\rho_m g K \beta_m |T_m - T_w|^{q}}{\mu}, \quad Ra = \frac{\rho_m g K \beta_m |T_m - T_w|^{q} r_0}{\alpha_m H}.
\end{equation}

Thus, Eqs.(2.1), (2.3) and (2.5) become
\begin{equation}
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0,
\end{equation}
\begin{equation}
u = \pm \left[ |\theta - R|^q - |R|^q \right],
\end{equation}
\begin{equation}
u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right)
\end{equation}
and the boundary conditions Eq.(2.6) take the form
\begin{equation}
v = 0, \quad \theta = 1 \quad \text{on} \quad r = 1, \quad x \geq 0,
\end{equation}
\begin{equation}
u \to 0, \quad \theta \to 0 \quad \text{as} \quad r \to \infty, \quad x \geq 0.
\end{equation}

We now look for a solution to Eqs.(2.9)-(2.12) of the form
\begin{equation}
\xi = 2 x^{1/2}, \quad \eta = (r^2 - 1)\xi, \quad \psi = (\xi/2)f(\xi, \eta), \quad \theta = \theta(\xi, \eta)
\end{equation}
where \( \psi \) is the stream function, which is defined in the usual way as
\begin{equation}
u = \frac{l}{r} \frac{\partial \psi}{\partial \eta}, \quad v = -\frac{l}{r} \frac{\partial \psi}{\partial x}.
\end{equation}

Using Eqs.(2.13) and (2.14), Eqs.(2.10)-(2.12) become
\begin{equation}\frac{\partial f}{\partial \eta} = \pm \left[ |\theta - R|^q - |R|^q \right], \quad (2.15)\end{equation}
where $R$ is the temperature parameter and is defined as

$$R = \frac{T_m - T_\infty}{T_w - T_\infty}$$

(2.17)

and these equations have to be solved subject to the boundary conditions

$$f = 0, \quad \theta = 1 \quad \text{on} \quad \eta = 0, \quad \xi \geq 0,$$

$$\frac{\partial f}{\partial \eta} \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty, \quad \xi \geq 0.$$

(2.18)

We notice that for small values of $\xi$, i.e., close to the leading edge of the cylinder, Eqs.(2.15) and (2.16) reduce to ordinary differential equations identical to those considered by Ramilison and Gebhart (1980), except a factor of $(\ell/2)$ in Eq.(2.16). On the other hand, it should be noticed that the role of the parameter $R$ is crucial in cold water transport. It replaces the prescribed temperatures $T_w$ and $T_\infty$, with respect to $T_m(s,p)$. From Eq.(2.15) it is seen that the vertical velocity is proportional to the buoyancy force. Therefore, the parameter $R$ indicates the local direction of the buoyancy force across the thermal boundary layer region and thus, also the direction of flow. A local flow reversal occurs across the boundary layer where the buoyancy force changes sign and it takes place at $\partial f / \partial \eta = 0$. The point where this occurs is denoted by $\eta_s$ and is determined from the equation

$$|\theta(\xi, \eta_s) - R|^q - |R|^q = 0.$$  

(2.19)

The only simplest solution to this equation is given by

$$\theta(\xi, \eta_s) = 2R.$$  

(2.20)

Therefore, for any given and admissible value of $R$, there may be a location $\eta_s$ where the vertical velocity is zero. The flow has opposite directions on opposite sides of this location in the thermal boundary layer. Since the temperature distribution $\theta(\xi, \eta)$ decreases monotonically from 1 to 0, the value of $R$ for which flow reversals occur is in the range $0 \leq R \leq 1/2$, see Eq.(2.20).

The physical parameter of interest is the local Nusselt number, $Nu$, which is given by

$$Nu / Ra_x^{1/2} = \frac{\partial \theta}{\partial \eta}(\xi_s, 0)$$  

(2.21)

where $Ra_x$ is the local Rayleigh number, which is defined as

$$Ra_x = \frac{\rho_m g \kappa \beta_m |T_m - T_\infty|^q x}{\alpha_m \mu}.$$  

(2.22)
Fig. 1. Effect of $R$ on the tangential velocity profiles. The solid lines are for upward flow, dotted lines – for downward flow.

Fig. 2. Effect of $R$ on the temperature profiles. The solid lines are for upward flow, dotted lines – for downward flow.
Fig. 3. Effect of $q$ on the tangential velocity profiles. The solid lines are for upward flow, dotted lines – for downward flow.

Fig. 4. Effect of $q$ on the temperature profiles. The solid lines are for upward flow, dotted lines – for downward flow.
Fig. 5. Development of the tangential velocity profiles. The solid lines are for upward flow, dotted lines – for downward flow.

Fig. 6. Development of the temperature profiles. The solid lines are for upward flow, dotted lines – for downward flow.
Fig. 7. Reversed flow conditions for various values of $R$.

Fig. 8. Effect of $R$ on the local Nusselt number. The solid lines are for upward flow, dotted lines for downward flow.
3. **Numerical results and discussion**

Equations (2.15) and (2.16) subject to the boundary conditions (2.18) have been integrated numerically for some values of the temperature parameter \( R \) in the range between \(-10\) and \(+10\) at some upstream coordinates \( \xi = 0.0 \) to \( 1.0 \) using the finite difference scheme developed by Blottner (1970). The values of \( q \) used are \( q = 1 \) (Boussinesq approximation) and \( q = 1.894816 \) (cold water approximation). The non-dimensional velocity \( \theta'(\xi, \eta) \) and non-dimensional temperature \( \theta(\xi, \eta) \) profiles are shown in Figs.1 to 7. Also, the variation of the local Nusselt number given by Eq.(2.21) is plotted in Figs.8 and 9. It is seen from these figures that the velocity profiles are higher for the upward flow (full lines) than those for the downward flow (broken lines) and vice versa for the temperature profiles. Further, Figs.3 and 4 show that the non-dimensional velocity profiles increase, while the non-dimensional temperature profiles decrease with an increase in the exponent \( q \). Since the temperature distribution \( \theta(\xi, \eta) \) decreases monotonically from 1 to 0, the values of \( R \) for which buoyancy and flow reversals occur are in the range \( 0.1 \leq R \leq 0.5 \) for \( q = 1.894816 \), see Fig.7. The present results are also compared in this figure with those of Ramilison and Gebhart (1980) for the corresponding problem of a vertical flat plate. An excellent agreement between these results can be noticed. It can be seen that for \( R = 0.45 \) and \( 0.5 \) the flow is completely reversed when \( \xi = 0 \) (flat plate with downward flow) and \( \xi = 1.0 \). However, for \( R > 0.5 \) there is no flow reverse when \( \xi = 1.0 \). On the other hand, Figs.8 and 9 show that the wall heat transfer increases with an increase of both \( R \) and \( q \). However, the local Nusselt number is greater for the downward flow than for the upward flow, and the increase in the local Nusselt number with \( \xi \) is almost linear.

4. **Conclusions**

Steady free convection boundary layer flow along a vertical circular cylinder embedded in a porous medium saturated with cold water has been studied numerically. Calculations extend over a wide range of
the temperature parameter $R$, including conditions both outside and inside the buoyancy force reversal region, $0 < R < 0.5$. In this region, buoyancy force and flow reversal arise, along with convective inversion. It is found that the variation of the density extremum across the velocity and thermal boundary layers in the porous medium has important effects on the flow and heat transfer characteristics. The present calculations, as well as those of Gebhart et al. (1983) for a horizontal flat plate in a porous medium filled with cold water, indicate that complicated mechanisms may arise and this suggests a need for an experimental investigation of this type of flow in a porous media saturated with cold water.

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Nomenclature

- $f$ - reduced stream function
- $g$ - magnitude of acceleration due to gravity
- $K$ - permeability of the porous medium
- $\text{Nu}_r$ - local Nusselt number
- $q$ - exponent in density Eq.(2.4)
- $r$ - non-dimensional radial coordinate
- $r_0$ - radius of cylinder
- $R$ - parameter defined in Eq.(2.17)
- $\text{Ra}$ - Rayleigh number
- $\text{Ra}_s$ - local Rayleigh number
- $s$ - salinity
- $T$ - fluid temperature
- $T_m$ - temperature at which maximum density occurs
- $u, v$ - non-dimensional velocity components along $x$- and $r$-directions, respectively
- $U_c$ - characteristic velocity
- $x$ - non-dimensional axial coordinate
- $\alpha_m$ - effective thermal diffusivity
- $\beta_m$ - coefficient of thermal expansion
- $\eta$ - pseudo-similarity variable
- $\eta_0$ - position of $\eta$ where the axial velocity is zero
- $\theta$ - non-dimensional temperature
- $\mu$ - viscosity of fluid
- $\rho$ - density
- $\rho_m$ - maximum density
- $\xi$ - non-dimensional stretched streamwise coordinate
- $\psi$ - non-dimensional stream function

Superscript

- * - dimensional variables

References

"Finite-difference methods of solution of the boundary-layer equations.
Free convection from a vertical cylinder embedded in a porous medium ...