Analytical Solutions for Flow of a Dusty Fluid Between Two Porous Flat Plates

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Equations governing flow of a dusty fluid between two porous flat plates with suction and injection are developed and closed-form solutions for the velocity profiles, displacement thicknesses, and skin friction coefficients for both phases are obtained. Graphical results of the exact solutions are presented and discussed.

Introduction

This paper deals with the two-dimensional, steady, laminar, fully developed flow of a dusty fluid between two parallel porous flat plates. The plates are infinitely long and separated by a fixed distance of $h1$ (h multiplied by 1, a constant) with the lower plate being coincident with the plane $y=0$. The flow takes place due to the action of a constant pressure gradient applied in the x-direction. Uniform fluid-phase suction and injection are imposed at the lower and upper plates, respectively. The fluid and particulate phases are both assumed incompressible.

In the present work both phases (the fluid phase and the particle cloud) are treated as continua. The basic assumption in the theoretical analysis of such a suspension is that the average properties of the particles are described in terms of continuous variables. Extensive work based on the continuum modeling of particulate (particle-fluid) suspensions has been reported (see, for instance, Marble, 1970; Di Giovanni and Lee, 1974; Ishii, 1975; and Drew, 1979, 1983).

The mathematical model employed in the present work represents a generalization of the original dusty-gas model (a model restricted for particulate suspensions having small volume fraction. See, for instance, Marble, 1970) by allowing for finite particulate volume fraction. In this case the particle-phase viscous effects are important.

In the absence of particle-phase viscous effects (small particulate volume fraction), it was reported by Chamkha (1992) that a difficulty exists as to the appropriate particle-phase boundary conditions that need to be used for this problem. The purpose of this paper is to obtain a closed-form solution for the problem described above for uniform and finite particle-phase volume fraction by applying slip boundary conditions familiar from rarefied gas dynamics on the particle phase. This allows one to explore the qualitative behavior brought about by changes in boundary conditions and various parameters of the system.

Governing Equations

Consider steady laminar flow of a suspension of solid spherical particles uniformly distributed in a continuous carrier fluid between two infinite parallel porous flat plates due to a constant applied pressure gradient. The governing equations for this investigation are based on the balance laws of mass and linear momentum for both phases. These are given by

$$\nabla \cdot ((1-\phi)\mathbf{V}) = 0, \quad (1a)$$

$$\nabla \cdot (\phi \mathbf{V}_p) = 0 \quad (1b)$$

$$\rho(1-\phi)\mathbf{V}\cdot\nabla\mathbf{V} = -\nabla((1-\phi)p) + \nabla\cdot(\mu(1-\phi)(\nabla\mathbf{V}+\nabla\mathbf{V}^T)) + \rho_p\phi(\mathbf{V}_p-\mathbf{V})/\tau, \quad (2a)$$

$$\rho_p\phi\mathbf{V}_p\cdot\nabla\mathbf{V}_p = \nabla\cdot(\mu_p\phi(\nabla\mathbf{V}_p+\nabla\mathbf{V}_p^T)) - \rho_p\phi(\mathbf{V}_p-\mathbf{V})/\tau \quad (2b)$$

where ∇ is the gradient operator, ϕ is the particle volume fraction, $\mathbf{V}$ is the fluid-phase velocity vector, $\mathbf{V}_p$ is the particle-phase velocity vector, ρ is the fluid-phase density, ρ_p is the particle-phase density, p is the fluid pressure, μ is the fluid-phase dynamic viscosity, τ is the momentum relaxation time (time needed for the relative velocity between the two phase to decrease e^{-1} of its original value), and μ_p is the particle-phase dynamic viscosity, and a superposed T denotes the transpose of a second order tensor. It can be seen from Eq. (2b) that the partial pressure contributed by the particle phase and gravity are neglected. This situation arises when inertia and drag dominate over gravity forces. This obtains when the velocity in the x-direction and the suction velocity are large compared to the particles settling velocity. The last term in Equation (2a) accounts for the interaction between the two phases and is based on Stoke's linear drag theory. In the present work, ϕ, ρ, ρ_p, μ, μ_p, and τ will all be treated as constants.

It is convenient to nondimensionalize the governing equations given earlier by using the following equations:

$$y = 1\eta, \quad \mathbf{V} = \mathbf{e}_x V_c F(\eta) - \mathbf{e}_y V_w,$$
$$\mathbf{V}_p = \mathbf{e}_x V_c F_p(\eta) - \mathbf{e}_y V_w, \quad dP/dx = -\mu V_c G/1^2 \quad (3)$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in the x and y directions, respectively, V_c is a characteristic velocity, and V_w is the suction (or injection) velocity and is a constant and positive. It can be noticed from Eqs. (2) that for steady-state and constant particulate volume fraction conditions the cross stream velocities for both phases have to be equal. The resulting nondimensional equations can be shown to be

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$$F'' + \mathrm{Re}_w F' - \kappa\alpha(F - F_p) + G = 0, \quad (4a)$$

$$r_\nu F_p'' + \mathrm{Re}_w F_p' + \alpha(F - F_p) = 0 \quad (4b)$$

where a prime denotes ordinary differentiation with respect to η and

$$\mathrm{Re}_w = V_w 1/\nu, \quad \kappa = \rho_p\phi/(\rho(1-\phi)), \quad \alpha = 1^2/(\tau\nu),$$
$$r_\nu = \nu_p/\nu\,(\nu = \mu/\rho, \quad \nu_p = \mu_p/\rho_p) \quad (5)$$

are the wall Reynolds number, the particle loading, the inverse Stokes number, and the viscosity ratio, respectively.

Four boundary conditions (two for the fluid phase and two for the particulate phase) are needed to solve Eqs. (4). No slip fluid boundary conditions will be used at the walls. That is

$$F(0) = 0, \quad F(h) = 0 \quad (6)$$

While the exact form of boundary conditions to be satisfied by a particulate phase at a surface is unknown at present and since a particle cloud may resemble a rarefied continuum, slip boundary conditions similar to those used in rarefied gas dynamics are used at the walls. These are

$$F_p'(0) = \omega F_p(0), \quad F_p'(h) = -\omega F_p(h) \quad (7)$$

where ω is a constant and positive, and the negative sign is used to make $F_p(h)$ positive since $F_p'(h)$ is negative. In general, ω would be a function of the coefficient of viscosity. No attempt, however, is made in the present work to relate ω to the internal properties of the suspension.

Of special interest are the fluid-phase volumetric flow rate, the particle-phase volumetric flow rate, the fluid-phase skin friction coefficient at the lower plate, and the particle-phase skin friction coefficient. These can be defined, respectively, as

$$Q = \int_0^h F(\eta)\,d\eta, \quad Q_p = \int_0^h F_p(\eta)\,d\eta,$$
$$C = F'(0), \quad C_p = r_\nu \kappa F_p'(0) \quad (8)$$

Results and Discussion

The governing equations developed above will be solved subject to the boundary conditions given earlier in closed form. This will be done next.

Solving for F in Eq. (4b), taking the appropriate derivatives, and then substituting into Eq. (4a) yield a fourth-order, linear, nonhomogeneous, ordinary differential equation in F_p. It can be written as

$$F^{iv} + \mathrm{Re}_w(1 + r_\nu)/r_\nu F_p''' + (\mathrm{Re}_w^2 - \alpha(1 + \kappa r_\nu))/r_\nu F_p''$$
$$- \mathrm{Re}_w\alpha(1+\kappa)/r_\nu F_p' = \alpha/r_\nu G \quad (9)$$

Without going into the details, it can be shown that

$$F_p = C_1\exp(m_1\eta) + C_2\exp(m_2\eta)$$
$$+ C_3\exp(m_3\eta) + A_1\eta + B_1 \quad (10)$$

where m_1, m_2, and m_3 are the roots of the equation

$$m^3 + P^*m^2 + qm + r = 0$$
$$P^* = \mathrm{Re}_w(1 + r_\nu)/r_\nu, \quad q = (\mathrm{Re}_w^2 - \alpha(1 + \kappa r_\nu))/r_\nu$$
$$r = -\mathrm{Re}_w\alpha(1+\kappa)/r_\nu, \quad A_1 = \alpha G/(r_\nu r) \quad (11)$$

It should be pointed out that the particle-phase axial velocity F_p is indirectly dependent on V_w through Re_w. The coefficients B_1, C_1, C_2, and C_3 are constants determined by the application of the boundary conditions. These can be shown to be

$$B_1 = \mathrm{Re}_w A_1/\alpha - C_1E - C_2H - C_3I,$$
$$C_1 = (-A_1(2+\omega h) - C_2K - C_3L)/J$$
$$C_2 = (R - C_3P_1)/\mathrm{O}, \quad C_3 = (Q_1\mathrm{O} - MR)/(N\mathrm{O} - MP_1) \quad (12)$$

where

$$E = 1 - m_1/\alpha(r_\nu m_1 + \mathrm{Re}_w), \quad H = 1 - m_2/\alpha(r_\nu m_2 + \mathrm{Re}_w)$$
$$I = 1 - m_3/\alpha(r_\nu m_3 + \mathrm{Re}_w),$$
$$J = (m_1 - \omega) + (m_1 + \omega)\exp(m_1h) \quad (13)$$

$$K = (m_2 - \omega) + (m_2 + \omega)\exp(m_2h),$$
$$L = (m_3 - \omega) + (m_3 + \omega)\exp(m_3h)$$
$$M = JH(1 - \exp(m_2h)) - KE(1 - \exp(m_1h))$$
$$N = IJ(1 - \exp(m_3h)) - LE(1 - \exp(m_1h)) \quad (14)$$

$$\mathrm{O} = K(\omega E + m_1 - \omega) - J(\omega H + m_2 - \omega),$$
$$P_1 = L(\omega E + m_1 - \omega) - J(\omega I + m_3 - \omega)$$
$$Q_1 = JA_1h + EA_1(2 + \omega h)(1 - \exp(m_1h))$$
$$R = -(2 + \omega h)A_1(\omega E + m_1 - \omega) - JA_1 - (\mathrm{Re}_w\omega/\alpha - 1) \quad (15)$$

With F_p known, Eq. (4a) can now be solved for F. The solution for F can be shown to be

$$F = C_1E(\exp(m_1\eta) - 1) + C_2H(\exp(m_2\eta)$$
$$-1) + C_3I(\exp(m_3\eta) - 1) + A_1\eta \quad (16)$$

The appropriate solutions for Q, Q_p, C, and C_p can be written, respectively, as

$$Q = (B_1 - \mathrm{Re}_w A_1/\alpha)h + A_1h^2/2 + C_1E(\exp(m_1h) - 1)/m_1$$
$$+ C_2\mathrm{H}(\exp(m_2h) - 1)/m_2 + C_3I(\exp(m_3h) - 1/m_3 \quad (17)$$

$$Q_p = B_1h + A_1h^2/2 + C_1(\exp(m_1h) - 1)/m_1$$
$$+ C_2(\exp(m_2h) - 1)/m_2 + C_3(\exp(m_3h) - 1)/m_3 \quad (18)$$

$$C = A_1C_1Em_1 + C_2Hm_2 + C_3Im_3 \quad (19)$$

$$C_p = \kappa r_\nu(A_1 + C_1m_1 + C_2m_2 + C_3m_3) \quad (20)$$

It is difficult to gain insight into the behavior of the physical properties of the problem under consideration from the form of the solutions reported above. For this reason, graphical results are obtained by numerically evaluating the exact solutions and will be presented below.

It should be mentioned that when r_ν was made very small during the numerical evaluation of the solutions, it was found that the corresponding solutions approached the inviscid case ($r_\nu = 0$) reported earlier by Chamkha (1992) under the appropriate conditions. This is a further evidence that the no slip condition on the particle phase used by Chamkha (1992) in the inviscid case is reasonable.

Figures 1 through 4 are obtained by numerically evaluating Eqs. (17) through (20). These figures are chosen from a variety of results to elucidate the features of the problem under consideration.

Figures 1 through 4 present the behavior of the fluid-phase volumetric flow rate Q, the particulate phase volumetric flow rate Q_p, the fluid-phase skin friction coefficient C, and the particulate-phase skin friction coefficient C_p for various values of the particle loading κ and the inverse Stokes number α, respectively. In these figures the dotted lines correspond to the equilibrium limit (where both phases are moving together with the same velocity) attained at large values of α ($\alpha \to \infty$). Increases in the values of α increase the momentum transfer between the two phases causing the fluid-phase volumetric flow rate to decrease and the particle-phase volumetric flow rate to increase until equilibrium between the two phases for large values of α is reached. This is evident from Figs. 1 and 2. It can be seen from Figs. 1 through 4 that as the particle loading (or the particle density) increases, Q, Q_p, and C decrease (since the fluid-phase velocity decreases and the particle phase is being dragged along by the fluid phase) while C_p increases (see definition of C_p) for all values of α shown. As apparent from Figs. 3 and 4, there is a minimum in C and a corresponding maximum in C_p at some particular value of α. This type of behavior is well known and often observed in relaxation type flows. It should be mentioned that when the present results were compared with the results associated with the inviscid

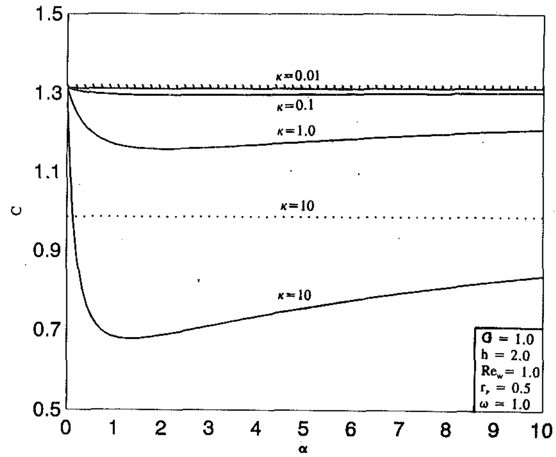

Fig. 1 Fluid-phase volumetric flow rate versus α

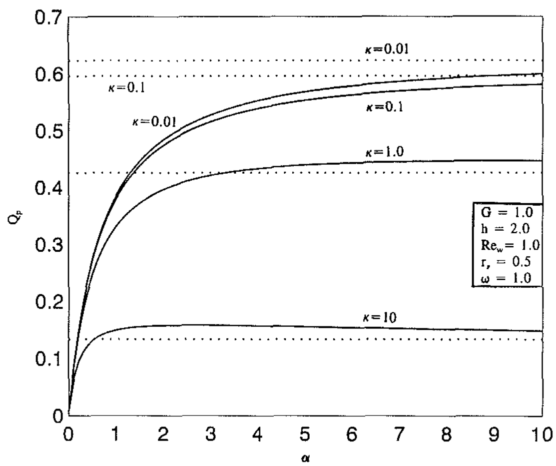

Fig. 2 Particle-phase volumetric flow rate versus α

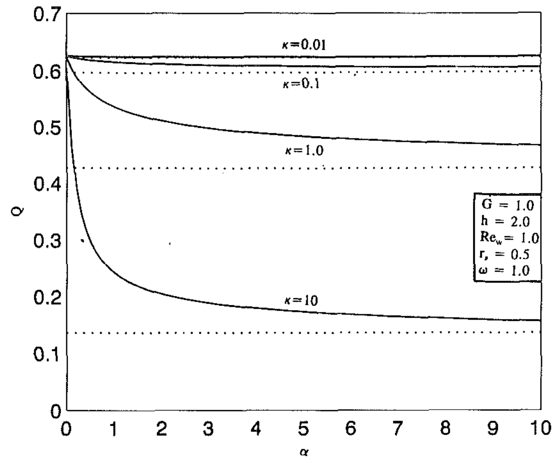

Fig. 3 Fluid-phase skin friction coefficient versus α

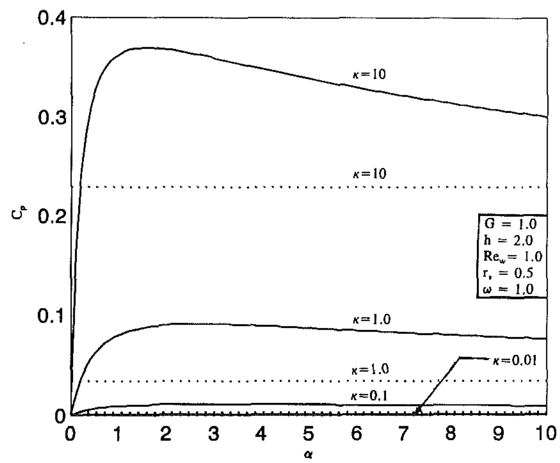

Fig. 4 Particle-phase skin friction coefficient versus α

particle phase case, significant decreases in the values of C and slight increases (for small values of α) followed by slight decreases (for large values of α) in the values of Q and Q_p were observed.

Conclusion

The problem of steady laminar fully developed flow of a particle-fluid suspension between two infinite parallel porous flat plates due to the action of a constant pressure gradient is solved in closed form. The particle-phase volume fraction is assumed finite and uniform. This assumption made it possible to investigate appropriate forms of particle-phase boundary conditions. Slip boundary conditions similar to those used in rarefied gas dynamics were utilized and appear to be reasonable. This provided a rational way in arriving at the appropriate boundary conditions for the case of small volume fraction where particle-phase viscous stresses are negligible. The influence of the particle loading on the volumetric flow rates and skin friction coefficients for both phases is presented graphically and discussed. It is hoped that the present model will be used for the investigation of different stress models and boundary conditions.

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