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MAGNETO-CONVECTION OF A TWO-FLUID FLOW THROUGH A VERTICAL CHANNEL

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ABSTRACT

The combined effect of viscous and Ohmic dissipations on magneto-convection of a two-fluid flow through a vertical channel with heated walls in the presence of applied electric field parallel to gravity and magnetic field normal to gravity with a heat source or sink is investigated. The coupled nonlinear equations are solved both analytically valid for small values of the product (Pr.Ec $=\varepsilon$) of the Prandtl number and Eckert number and numerically valid for small and large ε. These results are presented graphically for different values of the Hartmann number, Grashof number, viscosity ratio, width ratio, sources and sinks for open and short circuits. It is shown in the case of open circuit ($E \neq 0$) that, in contrast with the short circuit case (E=0), the effect of the magnetic field is to increase both the flow velocity and temperature in the channel.

KEYWORDS : Magneto-convection, source, sink, perturbation technique, finite-difference technique.

1. NOMENCLATURE

B_0 magnetic field strength

b ratio of the coefficients of thermal expansion (β_2/β_1)

C_p specific heat at constant pressure

E electric field loading parameter $\left(E_0/B_0\bar{u}_1\right)$

Ec Eckert number $\left(\bar{u}_1^2/C_P\left(T_{W_1}-T_{W_2}\right)\right)$

g acceleration due to gravity

h ratio of the widths of the two phases, (h_2/h_1)

h_i width of fluid i i=1,2

Gr Grashof number $\left(g\beta h_1^3\left(T_{W_1}-T_{W_2}\right)/\nu_1^2\right)$

K ratio of the thermal conductivities (K_1/K_2)

K_i thermal conductivity of fluid i, i=1,2

M Hartmann number $\left(B_0 h_1\sqrt{(\sigma_1/\mu_1)}\right)$

m ratio of viscosities μ_1/μ_2

n ratio of densities ρ_2/ρ_1

P non-dimensional pressure gradient $\left(\left(h_1^2/\mu_1\bar{u}_1\right)(\partial p/\partial x)\right)$

Pr Prandtl number $\left(\mu_1 C_p/K_1\right)$

Q_i dimensional heat generation or absorption of fluid i, i=1, 2

Re Reynolds number

σ ratio of electrical conductivities (σ_1/σ_2)

T_i temperature of fluid i, i=1, 2

T_{W1}, T_{W2} temperature at the boundaries

u_i velocity of fluid i, i=1, 2

$\overline{u_1}$ average velocity

x, y space coordinates

Greek Symbols

β_i coefficient of thermal expansion of fluid i, i=1, 2

σ_i electrical conductivity of fluid i, i=1, 2

ϕ dimensionless heat generation or absorption coefficient $\left(\phi=\dfrac{Qh}{k}\right)$

ρ_i density of fluid i, i=1, 2

ν kinematic viscosity

μ_i dynamic viscosity of fluid i, i=1, 2

ε product of Prandtl number and Eckert number (Pr.Ec)

θ non-dimensional temperature $((T-T_{W1})/(T_{W1}-T_{W2}))$

Subscripts

1, 2 refer to the quantities for Regions 1 and 2, respectively.

2. INTRODUCTION

Convective heat transfer in a vertical channel or pipe has been an important research topic for the last few decades because of its importance in several technological processes. Earliest studies on this subject were presented by Ostrach [1, 2], Rudraiah et al. [3] and Siegel [4]. Fletcher and Young [5] investigated convection in an electrically-conducting fluid in the presence of a magnetic field. Umavathi [6] analyzed the effect of magnetic field, electric field, and buoyancy in a vertical enclosure.

In recent years, there has been some theoretical and experimental work on the stratified laminar flow of two immiscible liquids in a horizontal pipe (Packham [7]). The interest in this configuration stems from the possibility of reducing the power required to pump oil in a pipeline by the suitable addition of water. The use of systems involving two-immiscible fluids covers a broad field, ranging from lubrication and cooling of equipment in metal-working processes to more delicate use as cosmetics. Immiscible fluids are also encountered at nearly every step of the petroleum production and recovery operations, viz., within underground porous media, at wellheads, in phase separators, in flotation units, in crude oil transport facilities, and at various stages of the refining process. The coal-fired magnetohydrodynamic (MHD) generator channel is subjected to an unusually severe thermal environment. Postlethwaite and Sluyter [8] presented an overview of the heat transfer problems associated with an MHD generator.

Shail [9] studied the problem of Hartmann flow of a conducting fluid in a horizontal channel of insulated plates with a layer of non-conducting fluid overlying a conducting fluid in two-fluid flow and predicted that the flow rate can be increased to the order of thirty percent for suitable values of ratios of viscosities, widths of the two fluids with appropriate Hartmann number. Very recently, Lohrasbi and Sahai [10] studied two-phase MHD flow and heat transfer in a parallel plate channel, with the fluid in one phase being electrically-conducting. Malashetty and Leela [11, 12] analyzed the Hartmann flow characteristics of two fluids in horizontal channel. The study of two-phase flow and heat transfer in an inclined channel has been studied by Malashetty and Umavathi [13] and Malashetty et al. [14, 15]. Chamkha [16] studied laminar flow of electrically conducting and heat generating or absorbing immiscible fluids in a parallel plate channel filled with porous media.

Many engineering applications, such as cooling of nuclear reactors, the design of MHD generators, cross–field accelerators, shock tubes and pumps, require natural convection with applied electric field. In addition, the use of electrically conducting and heat generating or absorbing fluids such as liquid metals as heat transfer in MHD power generation devices and nuclear engineering systems has created a growing interest in studying the influence of magnetic fields on such fluids and the subsequent effect on the efficiency of these devices and systems. The fluid mechanics and the heat transfer characteristics of the generator channel are significantly influenced by the presence of the magnetic field. The slag layer on the walls of the channel further complicated the problem. The present work is aimed toward modeling the physics of the problem as realistically as possible, but in the context of a simple geometry. The boundaries of the generator channel will be assumed to be infinitely long, parallel plates. The purpose of this study is to analyze the flow and heat transfer characteristics of the channel in the presence of magnetic and applied electric fields with internal heat generation or absorption. The resulting coupled nonlinear equations are solved analytically using a perturbation technique valid for small values of the product of Prandtl number and Eckert number and numerically using an implicit finite-difference technique, which involve large values of ε. The numerical solutions agree very well with the analytical solutions for small ε which reveals the accuracy of the numerical scheme.

3. MATHEMATICAL FORMULATION

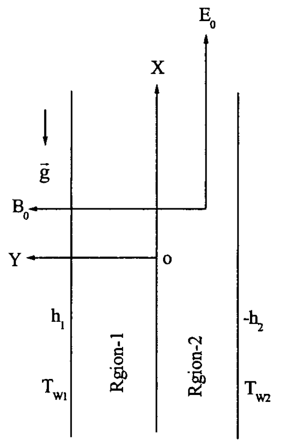

Figure 1. Physical configuration

The geometry under consideration illustrated in Fig.1, consists of two infinite parallel plates maintained at different constant temperatures, extending in the x and z directions. The region $0 \le y \le h_1$ is occupied by an electrically conducting fluid of density ρ_1, viscosity μ_1, electrical conductivity σ_1, thermal conductivity K_1 and the region $h_2 \le y \le 0$ is occupied by a different fluid of density ρ_2, viscosity μ_2, electrical conductivity σ_2 and thermal conductivity K_2. Both of the fluids are assumed to be heat generating or absorbing and have constant properties except the density in the buoyancy term in the momentum equations. A uniform magnetic field B_0 is applied across the channel normal to the plates and the uniform electric field E_0 is applied in the vertical direction. The fluid rises in the channel driven by buoyancy forces and retarded by magnetic forces. The transport properties of both fluids are assumed constant. We consider the fluids to be incompressible and the flow is steady, laminar and fully developed.

Under the assumptions stated above, the governing equations of momentum and energy become

Region-i

$$\mu_i \frac{d^2 u_i}{d y^2} + \rho_i g \beta_i (T_i - T_0) - \sigma_i (E_0 + B_0 u_i) B_0 = \frac{\partial P}{\partial x} \quad (1)$$

$$K_i \frac{d^2T_i}{dy^2} + \mu_i \left(\frac{du_i}{dy}\right)^2 + \sigma_i (E_0 + B_0 u_i)^2 \pm Q_i (T_i - T_0) = 0 \quad (2)$$

where the subscript i (i =1,2) indicates the quantities for Regions 1 and 2, respectively, u_i is the velocity in the x-direction, T_i is the temperature, T_0 is the ambient temperature, g is the acceleration due to gravity and β_i is the coefficient of thermal expansion.

To solve the above system of equations, one needs proper boundary and interface conditions. The physical hydrodynamic conditions are given by

$$\begin{aligned} &u_1(h_1) = 0 \\ &u_2(-h_2) = 0 \\ &u_1(0) = u_2(0) \\ &\mu_1 \frac{du_1}{dy_1}(0) = \mu_2 \frac{du_2}{dy_2}(0) \end{aligned} \quad (3)$$

The boundary and interface conditions on temperature are

$$\begin{aligned} &T_1(h_1) = T_{W1} \\ &T_2(-h_2) = T_{W2} \\ &T_1(0) = T_2(0) \\ &K_1 \frac{dT_1}{dy_1}(0) = K_2 \frac{dT_2}{dy_2}(0) \end{aligned} \quad (4)$$

The conditions on velocity represent the no-slip condition and continuity of velocity and shear stress across the interface. The conditions on temperature indicate that the plates are held at constant but different temperatures and continuity of heat and heat flux at the interface.

The basic equations and conditions are made dimensionless using the following transformations:

$$u_1^* = \frac{u_1}{\bar{u}_1} \quad u_2^* = \frac{u_2}{\bar{u}_1} \quad y_1^* = \frac{y_1}{h_1} \quad y_2^* = \frac{y_2}{h_2}$$
$$\theta_1 = \frac{(T_1 - T_{W_2})}{(T_{W_1} - T_{W_2})} \quad \theta_2 = \frac{(T_2 - T_{W_2})}{(T_{W_1} - T_{W_2})} \quad (5)$$

With the above non-dimensional quantities, the governing equations (1) and (2) become:

Region-1

$$\frac{d^2u_1}{dy^2} + \frac{Gr}{Re}\theta_1 - M^2(E + u_1) = P \quad (6)$$

$$\frac{d^2\theta_1}{dy^2} + Ec\,Pr\left(\frac{du_1}{dy}\right)^2 + Ec\,Pr\,M^2(E + u_1)^2 \pm \phi_1\theta_1 = 0 \quad (7)$$

Region-2

$$\frac{d^2u_2}{dy^2} + \frac{Gr}{Re} b\, m\, n\, h^2\theta_2 - m\,\sigma\, h^2M^2(E + u_2) = mh^2P \quad (8)$$

$$\frac{d^2\theta_2}{dy^2} + \frac{Ec\,Pr\,k}{m}\left(\frac{du_2}{dy}\right)^2 + Ec\,Pr\,k\,\sigma\, h^2M^2(E + u_2)^2 \pm \phi_2\theta_2 = 0 \quad (9)$$

The boundary and interface conditions (3) and (4) become:

$$\begin{aligned} &u_1(1) = 0 \\ &u_1(0) = u_2(0) \\ &u_2(-1) = 0 \\ &\frac{du_1}{dy} = \frac{1}{mh}\frac{du_2}{dy} \quad \text{at } y = 0 \end{aligned} \quad (10)$$

$$\begin{aligned} &\theta_1(0) = 1 \\ &\theta_1(0) = \theta_2(0) \\ &\theta_2(-1) = 0 \\ &\frac{d\theta_1}{dy} = \frac{1}{kh}\frac{d\theta_2}{dy} \quad \text{at } y = 0 \end{aligned} \quad (11)$$

where the asterisks have been dropped for simplicity.

4. ANALYTICAL SOLUTIONS

The governing equations (6)-(9) are to be solved subject to the boundary and interface conditions (10) and (11) for the velocity and temperature distributions. These equations are coupled and nonlinear because of the inclusion of the dissipation terms in the energy equation. In most of the practical problems the Eckert number, Ec is very small and is of order 10^{-5}. Accordingly the product Pr.Ec (= ε) is very small and hence this fact can be exploited to use the regular perturbation method with ε as the perturbation parameter. We assume the solutions in the form

$$(u_1, \theta_1) = (u_{10}, \theta_{10}) + \varepsilon(u_{11}, \theta_{11}) + \ldots \quad (12)$$
$$(u_2, \theta_2) = (u_{20}, \theta_{20}) + \varepsilon(u_{21}, \theta_{21}) + \ldots \quad (13)$$

where $u_{10}, u_{20}, \theta_{10}$, and θ_{20} are solutions for the case $\varepsilon = 0$ and $u_{11}, u_{21}, \theta_{11}$, and θ_{21} are corrections to the zeroth-order quantities. Substituting Eqs. (12) and (13) into Eqs. (6) to (11) and equating the like powers of ε to zero, yields the following set of equations:

Region-1

Zeroth-order equations

$$\frac{d^2u_{10}}{dy^2} + \frac{Gr}{Re}\theta_{10} - M^2(E + u_{10}) = P \quad (14)$$

$$\frac{d^2\theta_{10}}{dy^2} \pm \phi_1\theta_{10} = 0 \quad (15)$$

First-order equations

$$\frac{d^2u_{11}}{dy^2} + \frac{Gr}{Re}\theta_{11} - M^2u_{11} = 0 \quad (16)$$

$$\frac{d^2\theta_{11}}{dy^2}+\left(\frac{du_{10}}{dy}\right)^2+M^2(E+u_{10})^2\pm\phi_1\theta_{11}=0 \qquad (17)$$

Region-2

Zeroth-order equations

$$\frac{d^2u_{20}}{dy^2}+\frac{Gr}{Re}b\,m\,n\,h^2-m\,\sigma\,h^2M^2(E+u_{20})=mh^2P \qquad (18)$$

$$\frac{d^2\theta_{20}}{dy^2}\pm\phi_2\theta_{20}=0 \qquad (19)$$

First-order equations

$$\frac{d^2u_{21}}{dy^2}+\frac{Gr}{Re}bmnh^2\theta_{21}-m\sigma h^2M^2u_{21}=0 \qquad (20)$$

$$\frac{d^2\theta_{21}}{dy^2}+\frac{k}{m}\left(\frac{du_{20}}{dy}\right)^2+k\sigma h^2M^2(E+u_{20})^2\pm\phi_2\theta_{21}=0 \quad (21)$$

The corresponding boundary and interface conditions reduce to:

Zeroth-order conditions

$$\begin{aligned}&u_{10}(1)=0\\&u_{10}(0)=u_{20}(0)\\&u_{20}(-1)=0\\&\frac{du_{10}}{dy}=\frac{1}{mh}\frac{du_{20}}{dy}\quad \text{at } y=0\end{aligned} \qquad (22)$$

$$\begin{aligned}&\theta_{20}(-1)=0\\&\theta_{10}(1)=1\\&\theta_{10}(0)=\theta_{20}(0)\\&\frac{d\theta_{10}}{dy}=\frac{1}{kh}\frac{d\theta_{20}}{dy}\quad \text{at } y=0\end{aligned} \qquad (23)$$

First-order conditions

$$\begin{aligned}&u_{11}(1)=0\\&u_{11}(0)=u_{21}(0)\\&u_{21}(-1)=0\\&\frac{du_{11}}{dy}=\frac{1}{mh}\frac{du_{21}}{dy}\quad \text{at } y=0\end{aligned} \qquad (24)$$

$$\begin{aligned}&\theta_{11}(1)=0\\&\theta_{11}(0)=\theta_{21}(0)\\&\theta_{21}(-1)=0\\&\frac{d\theta_{11}}{dy}=\frac{1}{kh}\frac{d\theta_{21}}{dy}\quad \text{at } y=0\end{aligned} \qquad (25)$$

Equations (14)-(21) subject to the boundary and interface conditions (22)–(25) are solved analytically and these solutions are given below. In this case, the equations are coupled and linear and hence, closed-form solutions are found. The form of the solutions for θ_1 and θ_2 is different depending whether the sign before ϕ_1 and ϕ_2 is positive or negative. Therefore, two different solutions for θ_1 and θ_2 corresponding to the cases of heat-generating fluids $(+\phi_1>0,+\phi_2>0)$ and heat absorbing fluids $(-\phi_1<0,-\phi_2<0)$. are obtained. Obviously, cases where one of the fluids is heat generating and the other one is heat absorbing can be obtained in the same manner.

$$\theta_{10}=B_1\text{Cos}\sqrt{\phi_1}\,y+B_2\text{Sin}\sqrt{\phi_1}\,y \qquad (26)$$

$$\theta_{20}=C_1\text{Cos}\sqrt{\phi_2}\,y+C_2\text{Sin}\sqrt{\phi_2}\,y \qquad (27)$$

$$u_{10}=B_3e^{My}+B_4e^{-My}+l_1\text{Cos}\sqrt{\phi_1}\,y+l_2\text{Sin}\sqrt{\phi_1}\,y+l_3 \qquad (28)$$

$$u_{20}=C_3e^{\sqrt{A_3}y}+C_4e^{-\sqrt{A_3}y}+l_4\text{Cos}\sqrt{\phi_2}\,y+l_5\text{Sin}\sqrt{\phi_2}\,y+l_6 \qquad (29)$$

The solutions of the first-order perturbation equations (16), (17), (20), and (21) using boundary conditions (24) and (25) are

$$\begin{aligned}\theta_{11}=\;&B_5\text{Cos}\sqrt{\phi_1}\,y+B_6\text{Sin}\sqrt{\phi_1}\,y+r_{14}\text{Cos}2\sqrt{\phi_1}\,y+r_{15}\text{Si}\\&r_{16}e^{My}\text{Cos}\sqrt{\phi_1}\,y+r_{17}e^{My}\text{Sin}\sqrt{\phi_1}\,y+r_{18}e^{-My}\text{Cos}\sqrt{\phi}\\&r_{19}e^{-My}\text{Sin}\sqrt{\phi_1}\,y+r_{20}y\,\text{Cos}\sqrt{\phi_1}\,y+r_{21}y\text{Sin}\sqrt{\phi_1}\,y+\\&+r_{23}e^{-2My}+r_{24}e^{My}+r_{25}e^{-My}+r_{26}\end{aligned} \qquad (30)$$

$$\begin{aligned}\theta_{21}=\;&C_5\text{Cos}\sqrt{\phi_2}\,y+C_6\text{Sin}\sqrt{\phi_2}\,y+g_{14}\text{Cos}2\sqrt{\phi_2}\,y+g_{15}\text{Sir}\\&g_{16}e^{\sqrt{A_3}y}\text{Cos}\sqrt{\phi_2}\,y+g_{17}e^{\sqrt{A_3}y}\text{Sin}\sqrt{\phi_2}\,y+g_{18}e^{-\sqrt{A_3}y}\text{C}\\&g_{19}e^{-\sqrt{A_3}y}\text{Sin}\sqrt{\phi_2}\,y+g_{20}y\text{Cos}\sqrt{\phi_2}\,y+g_{21}y\text{Sin}\sqrt{\phi_2}\,y+\\&g_{23}e^{-2\sqrt{A_3}y}+g_{24}e^{\sqrt{A_3}y}+g_{25}e^{-\sqrt{A_3}Y}+g_{26}\end{aligned} \qquad (31)$$

$$\begin{aligned}u_{11}=\;&B_7e^{My}+B_8e^{-My}+l_7\text{Cos}2\sqrt{\phi_1}\,y+l_8\text{Sin}2\sqrt{\phi_1}\,y+l_9e^{My}\text{Cos}\sqrt{\phi_1}\,y+\\&l_{10}e^{My}\text{Sin}\sqrt{\phi_1}\,y+l_{11}e^{-My}\text{Cos}\sqrt{\phi_1}\,y+l_{12}e^{-My}\text{Sin}\sqrt{\phi_1}\,y+\\&l_{13}y\text{Cos}\sqrt{\phi_1}\,y+l_{14}y\text{Sin}\sqrt{\phi_1}\,y+l_{15}\text{Cos}\sqrt{\phi_1}\,y+l_{16}\text{Sin}\sqrt{\phi_1}\,y\\&+l_{17}e^{2My}+l_{18}e^{-2My}+l_{19}ye^{My}+l_{20}ye^{-My}+l_{21}\end{aligned} \qquad (32)$$

$$\begin{aligned}u_{21}=\;&C_7e^{\sqrt{A_3}y}+C_8e^{-\sqrt{A_3}y}+l_{22}\text{Cos}2\sqrt{\phi_2}\,y+l_{23}\text{Sin}2\sqrt{\phi_2}\,y+\\&l_{24}e^{\sqrt{A_3}y}\text{Cos}\sqrt{\phi_2}\,y+l_{35}e^{\sqrt{A_3}y}\text{Sin}\sqrt{\phi_2}\,y+l_{26}e^{-\sqrt{A_3}y}\text{Cos}\sqrt{\phi_2}\,y\\&l_{27}e^{-\sqrt{A_3}y}\text{Sin}\sqrt{\phi_2}\,y+l_{28}y\text{Cos}\sqrt{\phi_2}\,y+l_{29}y\text{Sin}\sqrt{\phi_2}\,y+l_{30}\text{Cos}\sqrt{}\\&l_{31}\text{Sin}\sqrt{\phi_2}\,y+l_{32}e^{2\sqrt{A_3}y}+l_{33}e^{-2\sqrt{A_3}y}+l_{34}ye^{\sqrt{A_3}y}+l_{35}ye^{-\sqrt{A}}\end{aligned} \qquad (33)$$

Heat-Absorption Case $(-\phi_1<0,-\phi_2<0)$. :

In a similar fashion as in the case of $+\phi_1>0$ and $+\phi_2>0$, the solutions for this case can be obtained. Solutions of the zeroth-order equations (14), (15), (18) and (19) using boundary conditions (22) and (23) are:

$$\theta_{10}=B\,\text{Cosh}\sqrt{\phi_1}\,y+B_2\,\text{Sinh}\sqrt{\phi_1}\,y \qquad (34)$$

$$\theta_{20}=C_1\,\text{Cosh}\sqrt{\phi_2}\,y+C_2\text{Sinh}\sqrt{\phi_2}\,y \qquad (35)$$

$$u_{10}=B_3\text{Cosh}M\,y+B_4\,\text{Sinh}M\,y+l_1\text{Cosh}\sqrt{\phi_1}\,y+l_2\text{Sinh}\sqrt{\phi_1}\,y+l_3 \qquad (36)$$

$$u_{20} = C_3 \operatorname{Cosh}\sqrt{A_3}\, y + C_4 \operatorname{Sinh}\sqrt{A_3}\, y + l_4 \operatorname{Cosh}\sqrt{\phi_2} + l_5 \operatorname{Sinh}\sqrt{\phi_2}\, y + l_6 \tag{37}$$

The solutions of the first-order equations (16), (17), (20), and (21) using boundary conditions (24) and (25) are

$$\begin{aligned}\theta_{11} = {} & B_5 \operatorname{Cos}\sqrt{\phi_1}\, y + B_6 \operatorname{Sin}\sqrt{\phi_1}\, y + r_{14} \operatorname{Cosh} 2My + r_{15} \operatorname{Sinh} 2My + \\ & r_{16} \operatorname{Cosh} 2\sqrt{\phi_1}\, y + r_{17} \operatorname{Sinh} 2\sqrt{\phi_1}\, y + r_{18} \operatorname{Cosh}\left(M + \sqrt{\phi_1}\right) y + \\ & r_{19} \operatorname{Cosh}\left(M - \sqrt{\phi_1}\right) y + r_{20} \operatorname{Sinh}\left(M + \sqrt{\phi_1}\right) y + r_{21} \operatorname{Sinh}\left(M - \sqrt{\phi_1}\right) y + \\ & r_{22}\, y \operatorname{Cosh}\sqrt{\phi_1}\, y + r_{23}\, y \operatorname{Sinh}\sqrt{\phi_1}\, y + r_{24} \operatorname{Cosh} My + r_{25} \operatorname{Sinh} My + r_{26}\end{aligned} \tag{38}$$

$$\begin{aligned}\theta_{21} = {} & C_5 \operatorname{Cosh}\sqrt{\Phi_2}\, y + C_6 \operatorname{Sinh}\sqrt{\Phi_2}\, y + g_{14} \operatorname{Cosh} 2\sqrt{A_3}\, y + \\ & g_{15} \operatorname{Sinh} 2\sqrt{A_3}\, y + g_6 \operatorname{Cosh} 2\sqrt{\Phi_2}\, y + g_{17} \operatorname{Sinh} 2\sqrt{\Phi_2}\, y + \\ & g_{18} \operatorname{Cosh}\left(\sqrt{A_3} + \sqrt{\Phi_2}\right) y + g_{19} \operatorname{Sinh}\left(\sqrt{A_3} + \sqrt{\Phi_2}\right) y + \\ & g_{20} \operatorname{Cosh}\left(\sqrt{A_3} - \sqrt{\Phi_2}\right) y + g_{21} \operatorname{Sinh}\left(\sqrt{A_3} - \sqrt{\Phi_2}\right) y + g_{22}\, y \operatorname{Cosh}\sqrt{\Phi} \\ & g_{23}\, y \operatorname{Sinh}\sqrt{\Phi_2}\, y + g_{24} \operatorname{Cosh}\sqrt{A_3}\, y + g_{25} \operatorname{Sinh}\sqrt{A_3}\, y + g_{26}\end{aligned} \tag{39}$$

$$\begin{aligned}u_{11} = {} & B_7 \operatorname{Cosh} My + B_8 \operatorname{Sinh} My + l_7 \operatorname{Cosh}\sqrt{\phi_1}\, y + \\ & l_8 \operatorname{Sinh}\sqrt{\phi_1}\, y + l_9 \operatorname{Cosh} 2My + l_{10} \operatorname{Sinh} 2My + \\ & l_{11} \operatorname{Cosh} 2\sqrt{\phi_1}\, y + l_{12} \operatorname{Sinh} 2\sqrt{\phi_1}\, y + l_{13} \operatorname{Cosh}\left(M + \sqrt{\phi_1}\right) y + \\ & l_{14} \operatorname{Cosh}\left(M - \sqrt{\phi_1}\right) y + l_{15} \operatorname{Sinh}\left(M + \sqrt{\phi_1}\right) y + \\ & l_{16} \operatorname{Sinh}\left(M - \sqrt{\phi_1}\right) y + l_{17}\, y \operatorname{Cosh} My + l_{18}\, y \operatorname{Sinh} My + \\ & l_{19}\, y \operatorname{Cosh}\sqrt{\phi_1}\, y + l_{20}\, y \operatorname{Sinh}\sqrt{\phi_1}\, y + l_{21}\end{aligned} \tag{40}$$

$$\begin{aligned}u_{21} = {} & C_7 \operatorname{Cosh}\sqrt{A_3}\, y + C_8 \operatorname{Sinh}\sqrt{A_3}\, y + l_{22} \operatorname{Cosh}\sqrt{\phi_2}\, y + \\ & l_{23} \operatorname{Sinh}\sqrt{\phi_2}\, y + l_{24} \operatorname{Cosh} 2\sqrt{A_3}\, y + l_{25} \operatorname{Sinh}\sqrt{\phi_2}\, y + \\ & l_{24} \operatorname{Cosh} 2\sqrt{A_3}\, y + l_{25} \operatorname{Sinh} 2\sqrt{A_3}\, y + l_{26} \operatorname{Cosh} 2\sqrt{\phi_2}\, y + \\ & l_{27} \operatorname{Sinh} 2\sqrt{\phi_2}\, y + l_{28} \operatorname{Cosh}\left(\sqrt{A_3} + \sqrt{\phi_2}\right) y + \\ & l_{29} \operatorname{Sinh}\left(\sqrt{A_3} + \sqrt{\phi_2}\right) y + l_{30} \operatorname{Cosh}\left(\sqrt{A_3} - \sqrt{\phi_2}\right) y + \\ & l_{31} \operatorname{Sinh}\left(\sqrt{A_3} - \sqrt{\phi_2}\right) y + l_{32}\, y \operatorname{Cosh}\sqrt{\phi_2}\, y + l_{23}\, y \operatorname{Sinh}\sqrt{\phi_2}\, y + \\ & l_{34} \operatorname{Cosh}\sqrt{A_3}\, y + l_{35} \operatorname{Sinh}\sqrt{A_3}\, y + l_{36}\end{aligned} \tag{41}$$

The expressions for the constants that appeared in the above equations are given in the Appendix. Apart from the velocity and temperature distributions in the channel, it is important to determine the rate of heat transfer between the plates and the fluids.

The rate of heat transfer through the channel wall to the fluid is given by

$$q = K \left(\frac{\partial T}{\partial y} \right)_{y = +h_1, -h_2} \tag{42}$$

A measure of the effectiveness of heat transfer is provided by the Nusselt number

$$Nu = \frac{(\text{rate of heat transfer})(\text{height of the channel})}{K(\text{temperature difference})} \tag{43}$$

This number at the left wall is given by

$$Nu_L = (1 + h) \left(\frac{d\theta}{dy} \right)_{y=1} \tag{44}$$

while at the right wall, it is given by

$$Nu_R = \left(1 + h^{-1}\right) \left(\frac{d\theta}{dy} \right)_{y=-1} \tag{45}$$

The variations of the left and right walls Nusselt numbers with different physical parameters for heat generation and absorption cases will be shown in tabular form in the next section.

Equations (26) to (41) for the velocity, temperature distributions and heat transfer coefficients are evaluated numerically and the results are discussed in the next section. Since the problem involves too many physical parameters, we fix some of them namely P=-5.0, Re=5.0, n=1.0, k=1.0, ε=0.1 and σ=2.0 for the sake of conciseness for all numerical computations and analyze the effect of other important non-dimensional parameters on flow and heat transfer characteristics. In each of the figures, all the other parameters expect the varying one are chosen from the set: (E, M, Gr, m, h, ϕ_1, ϕ_2) = (1.0, 4.0, 5.0, 1.0, 1.0, 1.0, 1.0) for heat generation and (E, M, Gr, m, h, ϕ_1, ϕ_2) = (1.0, 1.0, 5.0, 1.0, 1.0, 5.0, 5.0) for heat absorption. The velocity and temperature distributions are computed upto the first order and these are depicted in Figs. 2 to 9. A measure of the effectiveness of the heat transfer, namely the Nusselt number is computed for different values of M, Gr, ϕ, h and E and is presented in Tables 1 and 2. It should be noted that the obtained results are consistent with those previously reported by Chamkha [16] under the same conditions.

5. RESULTS AND DISCUSSION

In this section, the convective flow and heat transfer results for electrically conducting two-fluid flow through a vertical channel is discussed. The case $E = 0$ corresponds to the short circuit case and $E \neq 0$ corresponds to the open circuit case. The coupled nonlinear governing equations are solved using regular perturbation method with the product of Prandtl number and Eckert number as the perturbation parameter. Since the analytical solutions are valid only for small values of the perturbation parameter ε (=Pr.Ec), numerical solutions for large values of ε are also obtained using an implicit finite-difference methodology similar to that discussed by Blottner [17]. The details of the numerical method as to the step sizes employed and grid independence tests performed can be found in the paper by Chamkha [16]. The analytical solutions for the velocity and temperature profiles in the channel are computed for different values of governing parameters such as Hartmann number, Grashof number, viscosity ratio, widths ratio, heat generation and heat absorption parameters. The results from the analytical solutions are represented graphically in Figs. 2 to 9 along with those obtained by the numerical method. It is found that in all the cases presented excellent agreement between the analytical and numerical results is obtained.

Figure 2 presents the velocity distribution in both regions of the channel for different values of the Hartmann number M. It is seen that the effect of increasing the Hartmann number is to accelerate the flow in the positive direction of y for $E=-1$ and in the negative direction of y for $E=1$ exhibiting tunneling effect, namely a boundary layer nature near the boundaries and flattening in the middle of the channel.

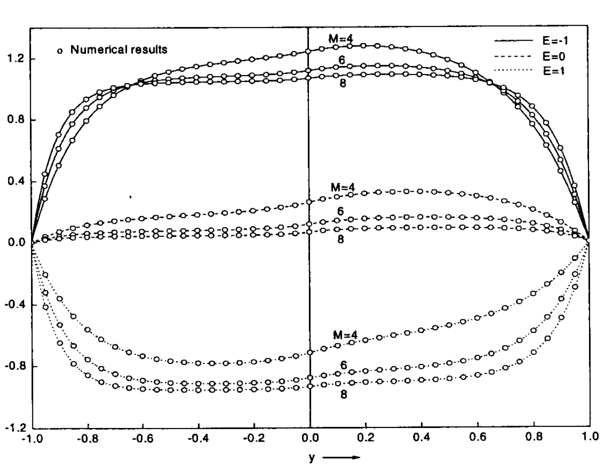

Fig. 2. Velocity profiles for different values of Hartmann number M.

Figures 3a (heat generation case) and 3b (heat absorption case) display the effect of the viscosity ratio m of the two fluids. It is found that the smaller the viscosity of the fluid in Region 1, the lower the flow field for open and short circuits. For the heat generation case (Fig. 3a) the direction of flow when $E>0$ is opposite to that when $E<0$ which can be effectively applied to problems for flow reversal situations.

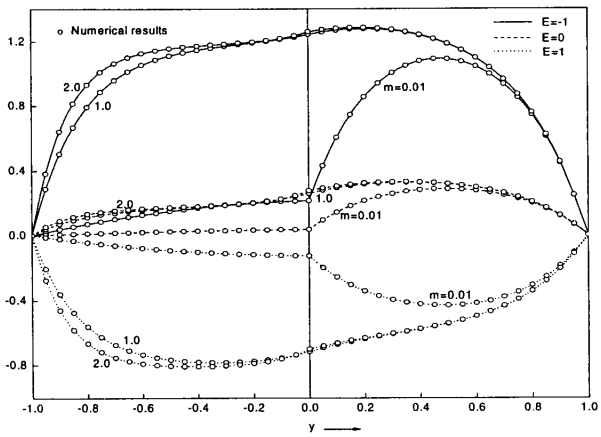

Fig. 3a. Velocity profiles for different values of the viscosity ratio m.

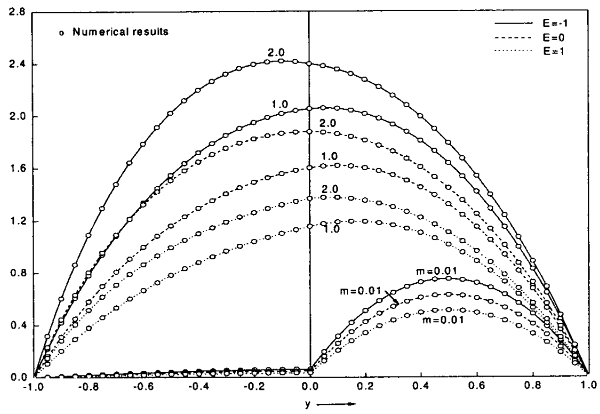

Fig. 3b. Velocity profiles for different values of the viscosity ratio m.

The effect of the ratio of the fluids widths h is displayed in Figs. 4a (heat generation case) and 4b (heat absorption case). It is predicted that the smaller the value of h, the slower the flow field. Similar to the effect of m, it is also observed that a flow reversal condition takes place for the heat generation case for E>0.

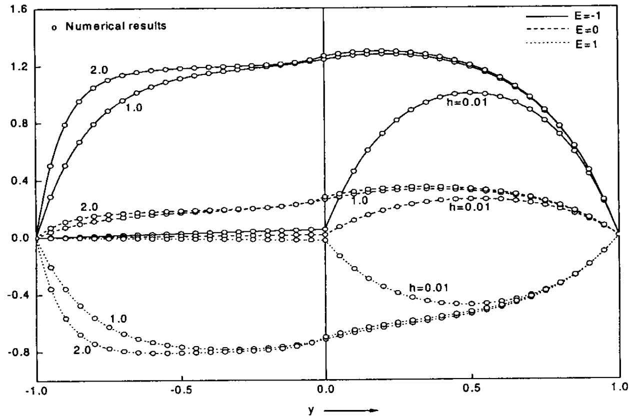

Fig. 4a. Velocity profiles for different values of the width ratio h.

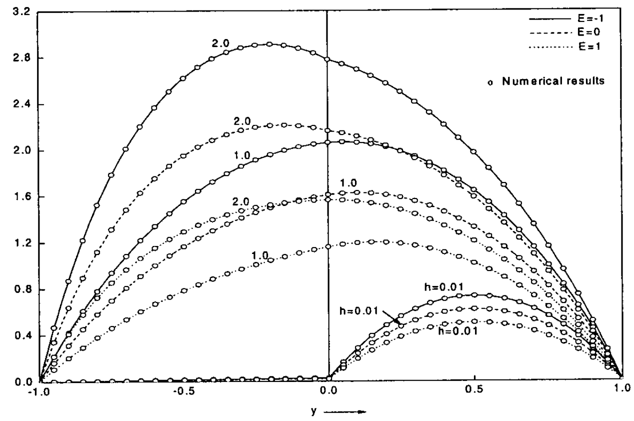

Fig. 4b. Velocity profiles for different values of the width ratio h.

Figures 5a (heat generation case) and 5b (heat absorption case) depict graphically the influence of the heat generation and heat absorption coefficients of both fluids ϕ_1 and ϕ_2 on the velocity field in the channel. For simplicity, it is taken $\phi_1=\phi_2=\phi$. In general, heat generation for both fluids has the tendency to increase the velocity in the channel and conversely heat absorption of both fluids decreases the velocity field for open and short circuits. However, it is interesting to note from Fig. 5a that for the parametric conditions used to obtain this figure the flow field in the channel increases as ϕ increases from 0.001 to 2.0 while it decreases as ϕ increases from 2.0 to 4.0. It is also observed from results not presented that for values of ϕ >4 there is a flow reversal condition in Region 2.

As expected, increasing the value of the Grashof number Gr causes increases in the buoyancy-induced flow in the channel resulting in increases in the velocity field in the channel. In addition, the effect of the Grashof number on the temperature profiles is the same as its effect on the velocity profiles. Namely, as Gr increases, the temperature field increases. This is also expected since as Gr increases, the convective currents in the channel increase. No figures are shown for the effect of Gr for the sake of simplicity.

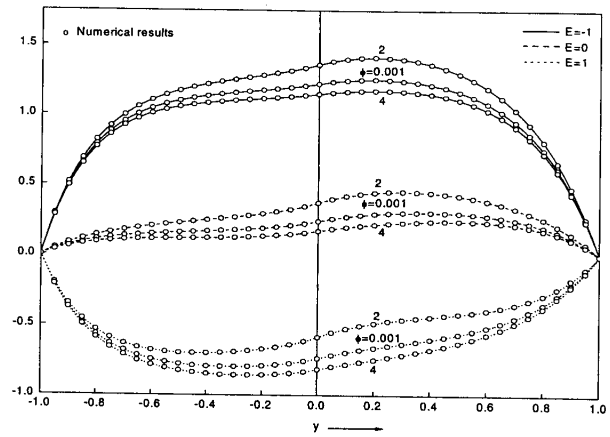

Fig. 5a. Velocity profiles for different values of heat generation coefficient ϕ.

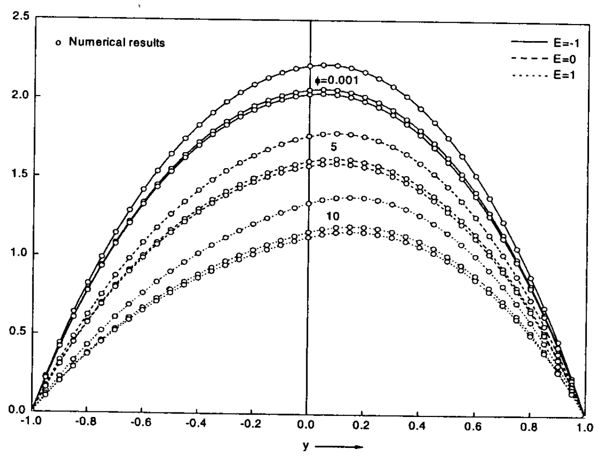

Fig. 5b. Velocity profiles for different values of heat absorption coefficient ϕ.

The effect of the fluids widths ratio h on the temperature profiles is presented in Figs. 6a (heat generation case) and 6b (heat absorption case). It is predicted that the temperature increases as h increases for both heat generation and absorption conditions and short and open circuits.

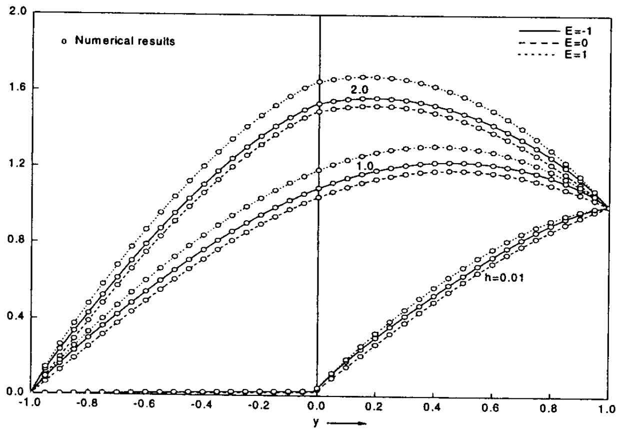

Fig. 6a. Temperature profiles for different values of the width ratio h.

Figures 7a and 7b depict that the effect of the heat generation and heat absorption coefficients on the temperature profiles is the same as that on the velocity profiles (Figs. 5a and 5b).

It is observed from other results not presented here for brevity that the effect of increasing the Hartmann number is to decrease the temperature for open and short circuits for heat generation conditions. However, for heat absorption conditions the temperature increases as M increases for $E \leq 0$ and decreases for $E>0$. Also, the effect of increasing the viscosity ratio of both fluids m is predicted to decrease (increase) the temperature for heat generation and $E \leq 0$ ($E>0$) and to increase the temperature for heat absorption for short and open circuits.

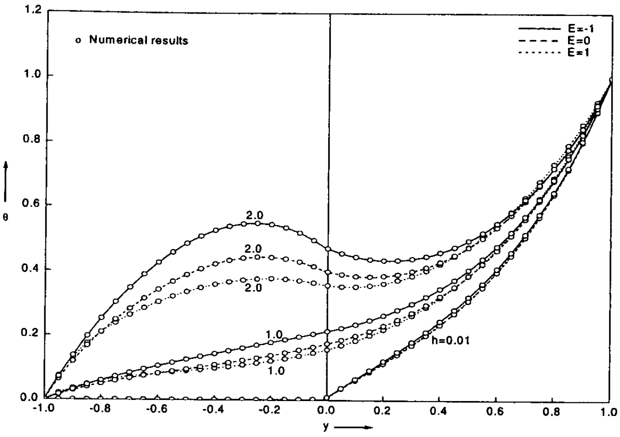

Fig. 6b. Temperature profiles for different values of the width ratio h.

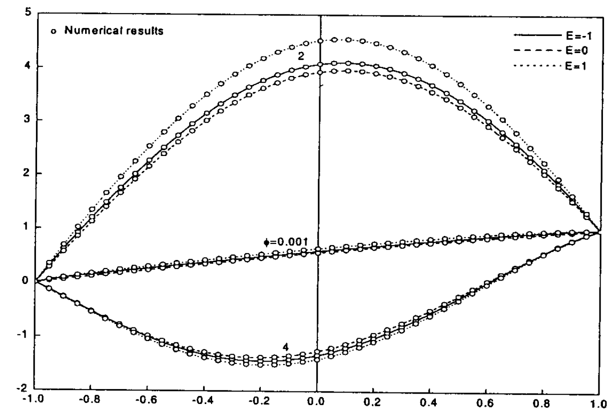

Fig. 7a. Temperature profiles for different values of heat generation coefficient ϕ.

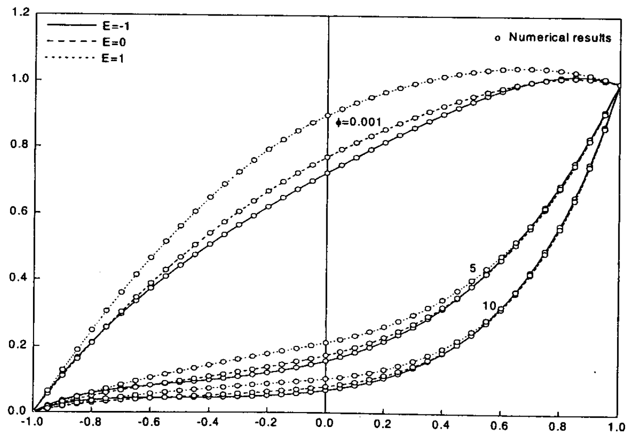

Fig. 7b. Temperature profiles for different values of heat absorption coefficient ϕ.

Figures 8 and 9 are obtained numerically by the finite-difference method. They illustrate the effect of the product $Pr.Ec = \varepsilon$ on the velocity and temperature profiles for both open and closed circuits and heat generation case. It is evident from these two figures that both the velocity and temperature fields in the channel increase as ε increases for open and closed circuits conditions. Also, it is seen that the degree of nonlinearity in the temperature profiles increase as ε increases.

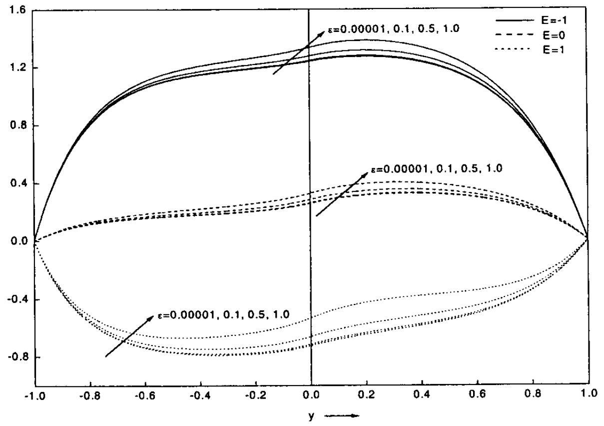

Fig. 8. Velocity profiles for different values of ε=PrEc.

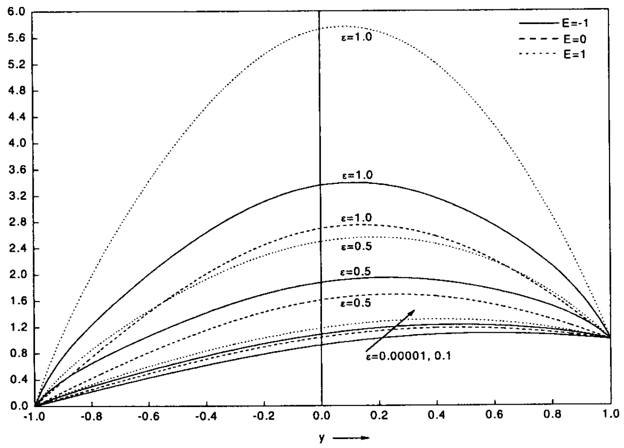

Fig. 9. Temperature profiles for different values of ε=PrEc.

The variations of the left and right wall Nusselt numbers (Nu_L and Nu_R) with different physical parameters and fixing the other parameters (as in the figures with E=1) are given in Tables 1 and 2 for heat generation and heat absorption conditions, respectively. We observe that increasing either of the strength of the magnetic field or the Grashof number increases the right wall Nusselt number and decreases the left wall Nusselt number for both heat generation and absorption conditions. However, while increasing the ratio of the widths of the two fluids also increases the right wall Nusselt number and decreases the left wall Nusselt number for heat generation conditions, it increases both Nu_L and Nu_R for heat absorption conditions. The increase in the value of the heat generation coefficient is observed to decrease Nu_L and to increase Nu_R up to $\phi = 2$ while the opposite effect occurs for values of $\phi > 2$ in Table 1. On the other hand, increasing the heat absorption coefficient increases Nu_L decreases Nu_R for all values of ϕ in Table 2. Finally, the effect of open electric circuit ($E \neq 0$) is seen to increase Nu_R and to decrease Nu_L for heat generation conditions and heat absorption conditions with the exception of E=1.

6. CONCLUSION

The problem of magneto-convection flow of two immiscible fluids through a vertical channel in the presence of an electric field and heat generation and absorption effects was investigated analytically for small viscous and magnetic dissipations and numerically for all conditions. Both fluids were assumed to be electrically conducting while the channel walls were assumed to be electrically insulating. Separate closed-form solutions for each fluid were obtained taking into consideration suitable interface matching conditions. The closed-form results were compared with numerical solutions based on the finite-difference methodology. These results were presented graphically for various values of the Hartmann number, fluids viscosities and widths ratios and heat generation or absorption coefficients for short and open electric circuits. It was found that the flow and heat transfer characteristics can be effectively controlled by the properties of both fluids as well as the presence or absence of electric field and heat generation or absorption effects.

M	Nu_L	Nu_R
4	-2.3257	3.9356
6	-2.4400	4.2617
8	-2.6885	4.7308
10	-2.9726	5.2421
12	-3.2641	5.7664
Gr		
1	-2.1922	3.8270
5	-2.3257	3.9356
10	-2.5391	4.1066
15	-2.8217	4.3316
$\phi_1 = \phi_2 = \phi$		
0.001	-0.0212	2.3459
1	-2.3257	3.9356
2	-13.4575	14.1845
3	11.1277	-11.3171
4	3.4786	-4.9545
H		
0.01	0.1458	3.0239
0.1	-0.0761	3.2076
0.5	-1.0520	3.5435
1.0	-2.3257	3.9356
2.0	-5.1591	4.8735
E		
-3.0	-9.4560	13.8667
-2.0	-4.9108	7.5349
-1.0	-2.2061	3.7666
0.0	-1.3434	2.5651
1.0	-2.3257	3.9356
2.0	-5.1587	7.8851
3.0	-9.8498	14.4230

Table 1. Variation of the Nusselt numbers (Heat Generation Case)

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M	Nu_L	Nu_R
1	3.8169	0.7087
4	3.7887	1.0743
6	3.4580	1.6120
8	3.1233	2.1617
10	2.7972	2.7113
Gr		
1	3.8768	0.6778
5	3.8169	0.7087
10	3.7335	0.7515
15	3.6392	0.7998
$\phi_1 = \phi_2 = \phi$		
0.001	-0.6091	2.6291
5	3.8169	0.7087
10	5.7935	0.4442
15	7.2429	0.3453
20	8.4376	0.2921
h		
0.01	2.1626	0.1286
0.1	2.3289	0.1539
0.5	3.0259	0.3274
1.0	3.8169	0.7087
2.0	5.1368	2.2543
E		
-3.0	2.6093	2.3053
-2.0	3.1771	1.4659
-1.0	3.5667	0.9209
0.0	3.7796	0.6689
1.0	3.8169	0.7087
2.0	3.6799	1.0390
3.0	3.3697	1.6585

Table 2. Variation of the Nusselt numbers (Heat Absorption Case)

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APPENDIX

Heat Absorption Case:

$$A_1 = \frac{Gr}{Re};\ A_2 = P + M^2E;\ A_3 = m\sigma h^2 M^2$$

$$A_4 = mh^2P + m\sigma h^2 M^2 E;\ A_5 = \frac{Grbmnh^2}{Re};$$

$$A_6 = \frac{k}{m};\ A_7 = k\sigma h^2 M^2;\ B_1 = \frac{1 - B_2 \operatorname{Sinh}\sqrt{\phi_1}}{\operatorname{Cosh}\sqrt{\phi_1}}$$

$$B_2 = \frac{\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_2}}{\sqrt{\phi_2}\operatorname{Sinh}\sqrt{\phi_1}\operatorname{Cosh}\sqrt{\phi_2} + \sqrt{\phi_1}\, k\, h \operatorname{Cosh}\sqrt{\phi_1}\operatorname{Sinh}\sqrt{\phi_2}}$$

$$B_3 = \frac{D_1 - B_4 \operatorname{Sinh}M}{\operatorname{Cosh}M};$$

$$B_4 = \frac{D_1 \operatorname{Cosh}\sqrt{A_3} - D_6 \operatorname{Cosh}M}{\operatorname{Sinh}M\operatorname{Cosh}\sqrt{A_3} + D_5 \operatorname{Sinh}\sqrt{A_3}\operatorname{Cosh}M}$$

$$B_5 = \frac{D_7 - B_6 \operatorname{Sinh}\sqrt{\phi_1}}{\operatorname{Cosh}\sqrt{\phi_1}}$$

$$B_6 = \frac{D_7\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_2} - D_9\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_1}}{\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_2}\operatorname{Sinh}\sqrt{\phi_1} + kh\sqrt{\phi_1}\operatorname{Sinh}\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_1}} - \frac{D_8\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_1} + D_{10}\operatorname{Sinh}\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_1}}{\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_2}\operatorname{Sinh}\sqrt{\phi_1} + kh\sqrt{\phi_1}\operatorname{Sinh}\sqrt{\phi_2}\operatorname{Cosh}\sqrt{\phi_1}}$$

$$B_7 = \frac{D_{11} - B_8 \operatorname{Sinh}M}{\operatorname{Cosh}M}$$

$$B_8 = \frac{D_{11}\sqrt{A_3}\operatorname{Cosh}\sqrt{A_3} - D_{13}\sqrt{A_3}\operatorname{Cosh}M}{\sqrt{A_3}\operatorname{Cosh}\sqrt{A_3}\operatorname{Sinh}M + mhM\operatorname{Sinh}\sqrt{A_3}\operatorname{Cosh}M} - \frac{D_{12}\sqrt{A_3}\operatorname{Cosh}\sqrt{A_3}\operatorname{Cosh}M + D_{14}\operatorname{Sinh}\sqrt{A_3}\operatorname{Cosh}M}{\sqrt{A_3}\operatorname{Cosh}\sqrt{A_3}\operatorname{Sinh}M + mhM\operatorname{Sinh}\sqrt{A_3}\operatorname{Cosh}M}$$

$$C_1 = \frac{\sqrt{\phi_1}\, k\, h\, \mathrm{Sinh}\sqrt{\phi_2}}{\sqrt{\phi_1}\, k\, h\, \mathrm{Cosh}\sqrt{\phi_1}\ \mathrm{Sinh}\sqrt{\phi_2} + \sqrt{\phi_2}\ \mathrm{Cosh}\sqrt{\phi_2}\ \mathrm{Sinh}\sqrt{\phi_1}}$$

$$C_2 = \frac{C_1\, \mathrm{Cosh}\sqrt{\phi_2}}{\mathrm{Sinh}\sqrt{\phi_2}} \qquad C_3 = B_3 - D_2$$

$$C_4 = \frac{C_3 \mathrm{Cosh}\sqrt{A_3} - D_3}{\mathrm{Sinh}\sqrt{A_3}} \qquad C_5 = B_5 - D_8$$

$$C_6 = \frac{C_5 \mathrm{Cosh}\sqrt{\phi_2} - D_9}{\mathrm{Sinh}\sqrt{\phi_2}} \qquad C_7 = B_7 - D_{12}$$

$$C_8 = \frac{C_7 \mathrm{Cosh}\sqrt{A_3} - D_{13}}{\mathrm{Sinh}\sqrt{A_3}}$$

$$D_1 = -\left[l_3 + l_1 \mathrm{Cosh}\sqrt{\phi_1} + l_2\ \mathrm{Sinh}\sqrt{\phi_1}\right];$$

$$D_2 = l_4 + l_6 - l_1 - l_3$$

$$D_3 = -\left[l_6 + l_4 \mathrm{Cosh}\sqrt{\phi_2} - l_5 \mathrm{Sinh}\sqrt{\phi_2}\right];$$

$$D_4 = l_5\sqrt{\phi_2} - m\,h\,l_2\sqrt{\phi_1}$$

$$D_5 = \frac{m\,h\,M}{\sqrt{A_3}};$$

$$D_6 = \frac{D_3\sqrt{A_3} - D_4 \mathrm{Sinh}\sqrt{A_3} + D_2\sqrt{A_3}\mathrm{Cosh}\sqrt{A_3}}{\sqrt{A_3}}$$

$$D_7 = -\begin{pmatrix} r_{14}\mathrm{Cosh}2M + r_{15}\mathrm{Sinh}2M + r_{16}\mathrm{Cosh}2\sqrt{\phi_1} + r_{17}\mathrm{Sinh}2\sqrt{\phi_1} + \\ r_{18}\mathrm{Cosh}\left(M+\sqrt{\phi_1}\right) + r_{19}\mathrm{Cosh}\left(M-\sqrt{\phi_1}\right) \\ + r_{20}\mathrm{Sinh}\left(M+\sqrt{\phi_1}\right) + r_{21}\mathrm{Sinh}\left(M-\sqrt{\phi_1}\right) + r_{22}\mathrm{Cosh}\sqrt{\phi_1} \\ + r_{23}\mathrm{Sinh}\sqrt{\phi_1} + r_{24}\mathrm{Cosh}M + r_{25}\mathrm{Sinh}M + r_{26} \end{pmatrix}$$

$$D_8 = g_{14} + g_{16} + g_{18} + g_{20} + g_{24} + g_{26} - r_{14} - r_{16} - r_{18}$$
$$- r_{19} - r_{24} - r_{26}$$

$$D_9 = -\begin{pmatrix} g_{14}\mathrm{Cosh}2\sqrt{A_3} - g_{15}\mathrm{Sinh}2\sqrt{A_3} + g_{16}\mathrm{Cosh}2\sqrt{\phi_2} \\ - g_{17}\mathrm{Sinh}2\sqrt{\phi_2} + g_{18}\mathrm{Cosh}\left(\sqrt{A_3}+\sqrt{\phi_2}\right) \\ - g_{19}\mathrm{Sinh}\left(\sqrt{A_3}+\sqrt{\phi_2}\right) + g_{20}\mathrm{Cosh}\left(\sqrt{A_3}-\sqrt{\phi_2}\right) \\ - g_{21}\mathrm{Sinh}\left(\sqrt{A_3}-\sqrt{\phi_2}\right) - g_{22}\mathrm{Cosh}\sqrt{\phi_2} \\ + g_{23}\mathrm{Sinh}\sqrt{\phi_2} + g_{24}\mathrm{Cosh}\sqrt{A_3} - g_{25}\mathrm{Sinh}\sqrt{A_3} + g_{26} \end{pmatrix}$$

$$D_{10} = g_{15}2\sqrt{A_3} + g_{19}\left(\sqrt{A_3}+\sqrt{\phi_2}\right) + g_{21}\left(\sqrt{A_3}-\sqrt{\phi_2}\right)$$
$$+ g_{17}2\sqrt{\phi_2} + g_{25}\sqrt{A_3} + g_{22} - 2Mkhr_{15}$$
$$- 2\sqrt{\phi_1}khr_{17} - khr_{20}\left(M+\sqrt{\phi_1}\right) - khr_{21}\left(M-\sqrt{\phi_1}\right)$$
$$- khMr_{25} - khr_{22}$$

$$D_{12} = l_{22} + l_{24} + l_{26} + l_{28} + l_{30} + l_{34} + l_{36} - l_7 - l_9 - l_{11}$$
$$- l_{13} - l_{14} - l_{21}$$

$$D_{11} = -\begin{pmatrix} l_7\mathrm{Cosh}\sqrt{\phi_1} + l_8\mathrm{Sinh}\sqrt{\phi_1} + l_9\mathrm{Cosh}2M + l_{10}\mathrm{Sinh}2M \\ + l_{11}\mathrm{Cosh}2\sqrt{\phi_1} + l_{12}\mathrm{Sinh}2\sqrt{\phi_1} + l_{13}\mathrm{Cosh}\left(M+\sqrt{\phi_1}\right) \\ + l_{14}\mathrm{Cosh}\left(M-\sqrt{\phi_1}\right) + l_{15}\mathrm{Sinh}\left(M+\sqrt{\phi_1}\right) \\ + l_{16}\mathrm{Sinh}\left(M-\sqrt{\phi_1}\right) + l_{17}\mathrm{Cosh}M + l_{18}\mathrm{Sinh}M \\ + l_{19}\mathrm{Cosh}\sqrt{\phi_1} + l_{20}\mathrm{Sinh}\sqrt{\phi_1} + l_{21} \end{pmatrix}$$

$$D_{13} = -\begin{pmatrix} l_{22}\mathrm{Cosh}\sqrt{\phi_2} - l_{23}\mathrm{Sinh}\sqrt{\phi_2} + l_{24}\mathrm{Cos}2\sqrt{A_3} - l_{25}\mathrm{Sinh}\sqrt{A_3} \\ + l_{26}\mathrm{Cosh}2\sqrt{\phi_2} - l_{27}\mathrm{Sinh}2\sqrt{\phi_2} + l_{28}\mathrm{Cosh}\left(\sqrt{A_3}+\sqrt{\phi_2}\right) \\ - l_{29}\mathrm{Sinh}\left(\sqrt{A_3}+\sqrt{\phi_2}\right) + l_{30}\mathrm{Cosh}\left(\sqrt{A_3}-\sqrt{\phi_2}\right) \\ - l_{31}\mathrm{Sinh}\left(\sqrt{A_3}-\sqrt{\phi_2}\right) - l_{32}\mathrm{Cosh}\sqrt{\phi_2} + l_{33}\mathrm{Sinh}\sqrt{\phi_2} \\ + l_{34}\mathrm{Cosh}\sqrt{A_3} - l_{35}\mathrm{Sinh}\sqrt{A_3} + l_{36} \end{pmatrix}$$

$$D_{14} = l_{23}\sqrt{\phi_2} + l_{25}2\sqrt{A_3} + l_{29}\left(\sqrt{A_3}+\sqrt{\phi_2}\right)$$
$$+ l_{31}\left(\sqrt{A_3}-\sqrt{\phi_2}\right) + 2\sqrt{\phi_2}l_{27} + l_{35}\sqrt{A_3}$$
$$+ l_{32} - mhl_8\sqrt{\phi_1} - 2mhMl_{10} - 2mh\sqrt{\phi_1}l_{12}$$
$$- mhl_{15}\left(M+\sqrt{\phi_1}\right) - mhl_{16}\left(M-\sqrt{\phi_1}\right) - mhl_{17} - mhl_{19}$$

$$g_1 = \frac{-\left[A_6A_3C_3^2 + A_7C_4^2 + A_6A_3C_4^2 + A_7C_3^2\right]}{2}$$

$$g_2 = \frac{-\left[A_6\, l_4^2\, \phi_2 + A_7 l_5^2 + A_6 l_5^2\, \phi_2 + A_7\, l_4^2\right]}{2}$$

$$g_3 = -C_3C_4\left(A_6A_3 + A_7\right)$$

$$g_4 = -\begin{bmatrix} A_6C_3l_4\sqrt{A_3}\sqrt{\phi_2} + A_7C_4l_5 + \\ A_6C_4\,l_5\sqrt{A_3}\sqrt{\phi_2} + A_7C_3\,l_4 \end{bmatrix}$$

$$g_5 = A_6C_3l_4\sqrt{A_3}\sqrt{\phi_2} + A_7C_4l_5$$
$$- A_6C_4\,l_5\sqrt{A_3}\sqrt{\phi_2} - A_7C_3\,l_4$$

$$g_6 = -\begin{pmatrix} A_6C_3l_5\sqrt{A_3}\sqrt{\phi_2} + A_7C_4l_4 + \\ A_6C_4l_4\sqrt{A_3}\sqrt{\phi_2} + A_7C_3l_5 \end{pmatrix}$$

$$g_7 = \begin{pmatrix} A_6C_4l_4\sqrt{A_3}\sqrt{\phi_2} + A_7C_3l_5 - \\ A_6C_3l_5\sqrt{A_3}\sqrt{\phi_2} - A_7C_4l_4 \end{pmatrix}$$

$$g_8 = -\left(A_6l_4l_5\,\phi_2 + A_7l_4l_5\right);$$

$$g_9 = -2\left(A_7C_3l_6 + EA_7C_3\right)$$

$$g_{10} = -2\left(A_7C_4l_6 + EA_7C_4\right);$$

$$g_{11} = -2\left(A_7l_4l_6 + E\,A_7l_4\right)$$

$$g_{12} = -2\left(A_7\,l_5\,l_6 + E\,A_7l_5\right);$$

$$g_{13} = S_1 - S_2 + S_3 - S_4 - S_5$$

$$g_{14} = \frac{g_1}{4A_3 - \phi_2}; \qquad g_{15} = \frac{g_3}{4A_3 - \phi_2};$$

$$g_{16} = \frac{g_2}{3\phi_2} \qquad g_{17} = \frac{g_8}{3\phi_2}$$

$$g_{18}=\frac{g_4}{A_3+2\sqrt{A_3}\sqrt{\phi_2}};\quad g_{19}=\frac{g_6}{A_3+2\sqrt{A_3}\sqrt{\phi_2}};$$

$$g_{20}=\frac{g_5}{A_3-2\sqrt{A_3}\sqrt{\phi_2}};\quad g_{21}=\frac{g_7}{A_3-2\sqrt{A_3}\sqrt{\phi_2}};$$

$$g_{22}=\frac{g_{12}}{2\sqrt{\phi_2}};\quad g_{23}=\frac{g_{11}}{2\sqrt{\phi_2}};\quad g_{24}=\frac{g_9}{A_3-\phi_2};$$

$$g_{25}=\frac{g_{10}}{A_3-\phi_2};\quad g_{26}=\frac{-g_{13}}{\phi_2}$$

$$l_1=\frac{-A_1B_1}{\phi_1-M^2};\quad l_2=\frac{-A_2B_2}{\phi_1-M^2}$$

$$l_3=\frac{-A_2}{M^2};\quad l_4=\frac{-A_5C_1}{\phi_2-A_3}$$

$$l_5=\frac{-A_5C_2}{\phi_2-A_3};\quad l_6=\frac{-A_4}{A_3};$$

$$l_7=\frac{2A_1r_{23}\sqrt{\phi_1}-A_1B_5\left(\phi_1-M^2\right)}{\left(\phi_1-M^2\right)^2}$$

$$l_8=\frac{2A_1r_{22}\sqrt{\phi_1}-A_1B_6\left(\phi_1-M^2\right)}{\left(\phi_1-M^2\right)^2}$$

$$l_9=\frac{-A_1r_{14}}{3M^2}\quad l_{10}=\frac{-A_1r_{15}}{3M^2}\quad l_{11}=\frac{-A_1r_{16}}{4\phi_1-M^2}$$

$$l_{12}=\frac{-A_1r_{17}}{4\phi_1-M^2};\ l_{13}=\frac{-A_1r_{18}}{\phi_1+2M\sqrt{\phi_1}};\ l_{14}=\frac{-A_1r_{19}}{\phi_1-2M\sqrt{\phi_1}}$$

$$l_{15}=\frac{-A_1r_{20}}{\phi_1+2M\sqrt{\phi_1}};\quad l_{16}=\frac{-A_1r_{21}}{\phi_1-2M\sqrt{\phi_1}};$$

$$l_{17}=\frac{-A_1r_{25}}{2M}\quad l_{18}=\frac{-A_1r_{24}}{2M};$$

$$l_{19}=\frac{-A_1r_{22}}{\phi_1-M^2};\ l_{20}=\frac{-A_1r_{23}}{\phi_1-M^2}\quad l_{21}=\frac{A_1r_{26}}{M^2};$$

$$l_{22}=\frac{\left(2A_5g_{23}\sqrt{\phi_2}-A_5C_5\left(\phi_2-A_3\right)\right)}{\left(\phi_2-A_3\right)^2}$$

$$l_{23}=\frac{\left(2A_5g_{22}\sqrt{\phi_2}-A_5C_6\left(\phi_2-A_3\right)\right)}{\left(\phi_2-A_3\right)^2};$$

$$l_{24}=\frac{-A_5g_{14}}{3A_3};\ l_{25}=\frac{-A_5g_{15}}{3A_3};\quad l_{26}=\frac{-A_5g_{16}}{4\phi_2-A_3};$$

$$l_{27}=\frac{-A_5g_{17}}{4\phi_2-A_3};\quad l_{28}=\frac{-A_5g_{18}}{\phi_2+2\sqrt{A_3}\sqrt{\phi_2}};$$

$$l_{29}=\frac{-A_5g_{19}}{\phi_2+2\sqrt{A_3}\sqrt{\phi_2}};\quad l_{30}=\frac{-A_5g_{20}}{\phi_2-2\sqrt{A_3}\sqrt{\phi_2}}$$

$$l_{31}=\frac{-A_5g_{21}}{\phi_2-2\sqrt{A_3}\sqrt{\phi_2}};\quad l_{32}=\frac{-A_5g_{22}}{\phi_2-A_3};$$

$$l_{33}=\frac{-A_5g_{23}}{\phi_2-A_3};\ l_{34}=\frac{-A_5g_{24}}{2\sqrt{A_3}};\quad l_{35}=\frac{-A_5g_{25}}{2\sqrt{A_3}};$$

$$l_{36}=\frac{A_5g_{26}}{A_3};\ r_1=M^2\left(B_3^2+B_4^2\right);$$

$$r_2=2B_3B_4M^2;\ r_3=\frac{l_1^2\phi_1+M^2l_2^2+l_2^2\phi_1+M^2l_1^2}{2};$$

$$r_4=l_1l_2\phi_1+M^2l_1l_2$$

$$r_5=B_3Ml_1\sqrt{\phi_1}+M^2B_4l_2+B_4Ml_2\sqrt{\phi_1}+M^2B_3l_1$$

$$r_6=B_4Ml_2\sqrt{\phi_1}+M^2B_3l_1-B_3Ml_1\sqrt{\phi_1}-M^2B_4l_2$$

$$r_7=B_3Ml_2\sqrt{\phi_1}+M^2B_4l_1+B_4Ml_1\sqrt{\phi_1}+M^2B_3l_2$$

$$r_8=B_3Ml_2\sqrt{\phi_1}+M^2B_4l_1-B_4Ml_1\sqrt{\phi_1}-M^2B_3l_2$$

$$r_9=2M^2B_3\left(l_3+E\right);\quad r_{10}=2M^2B_4\left(l_3+E\right);$$

$$r_{11}=2M^2l_1\left(l_3+E\right);\quad r_{12}=2M^2l_2\left(l_3+E\right);$$

$$r_{13}=\frac{2M^2E^2+2M^2l_3^2+4EM^2{l_3}^2l_2^2}{2}$$

$$+\frac{l_2^2\phi_1+M^2l_1^2-l_1^2\phi_1-M}{2}$$

$$r_{14}=\frac{-r_1}{4M^2-\phi_1};\ r_{15}=\frac{-r_2}{4M^2-\phi_1};\ r_{16}=\frac{-r_3}{3\phi_1}$$

$$r_{17}=\frac{-r_4}{3\phi_1};\quad r_{18}=\frac{-r_5}{M^2+2M\sqrt{\phi_1}};$$

$$r_{19}=\frac{-r_6}{M^2-2M\sqrt{\phi_1}}\quad r_{20}=\frac{-r_7}{M^2+2M\sqrt{\phi_1}};$$

$$r_{21}=\frac{-r_8}{M^2-2M\sqrt{\phi_1}};\quad r_{22}=\frac{-r_{12}}{2\sqrt{\phi_1}}$$

$$r_{23}=\frac{-r_{11}}{2\sqrt{\phi_1}};\quad r_{24}=\frac{-r_9}{M^2-\phi_1};$$

$$r_{25}=\frac{-r_{10}}{M^2-\phi_1};\quad r_{26}=\frac{r_{13}}{\phi_1}$$

$$S_1=\frac{\left(A_6A_3C_3^2+A_7C_4^2\right)}{2};\ S_2=\frac{\left(A_6A_3C_4^2+A_7C_3^2\right)}{2};$$

$$S_3=\frac{\left(A_6l_4^2\phi_2+A_7l_5^2\right)}{2};\ S_4=\frac{\left(A_6l_5^2\phi_2+A_7l_4^2\right)}{2};$$

$$S_5=A_7\left(E+l_6\right)^2$$

Heat Generation Case:

$$A_1=\frac{Gr}{Re};\ A_2=P+M^2E;\ A_3=m\sigma h^2M^2$$

$$A_4=mh^2P+m\sigma h^2M^2E;\ A_5=\frac{Grbm\sigma h^2}{Re};\ A_6=\frac{k}{m}$$

$$A_7 = k\sigma h^2 M^2; \quad B_1 = \frac{1 - B_2 \mathrm{Sin}\sqrt{\phi_1}}{\mathrm{Cos}\sqrt{\phi_1}};$$

$$B_2 = \frac{\sqrt{\phi_2}\mathrm{Cos}\sqrt{\phi_2}}{\sqrt{\phi_2}\mathrm{Sin}\sqrt{\phi_1}\mathrm{Cos}\sqrt{\phi_2} + \sqrt{\phi_1}kh\,\mathrm{Sin}\sqrt{\phi_2}\mathrm{Cos}\sqrt{\phi_1}}$$

$$B_3 = \frac{D_1 - B_4 e^{-M}}{e^M};$$

$$B_4 = \frac{\sqrt{A_3}\left(1+e^{-2\sqrt{A_3}}\right)D_5 - \left(1-e^{-2\sqrt{A_3}}\right)D_6}{\sqrt{A_3}\left(1+e^{-2\sqrt{A_3}}\right)\left(1-e^{-2M}\right) + mhM\left(1+e^{-2M}\right)\left(1-e^{-2\sqrt{A_3}}\right)}$$

$$B_5 = \frac{D_7 - B_6 \mathrm{Sin}\sqrt{\phi_1}}{\mathrm{Cos}\sqrt{\phi_1}};$$

$$B_6 = \frac{\sqrt{\phi_2}D_7\mathrm{Cos}\sqrt{\phi_2} - \sqrt{\phi_2}D_9\mathrm{Cos}\sqrt{\phi_1}}{\sqrt{\phi_2}\mathrm{Sin}\sqrt{\phi_1}\,\mathrm{Cos}\sqrt{\phi_2} + kh\sqrt{\phi_1}\mathrm{Sin}\sqrt{\phi_2}\,\mathrm{Cos}\sqrt{\phi_1}} - \frac{\sqrt{\phi_2}D_8\mathrm{Cos}\sqrt{\phi_2}\,\mathrm{Cos}\sqrt{\phi_1} + D_{10}\mathrm{Sin}\sqrt{\phi_2}\,\mathrm{Cos}\sqrt{\phi_1}}{\sqrt{\phi_2}\mathrm{Sin}\sqrt{\phi_1}\,\mathrm{Cos}\sqrt{\phi_2} + kh\sqrt{\phi_1}\mathrm{Sin}\sqrt{\phi_2}\,\mathrm{Cos}\sqrt{\phi_1}}$$

$$C_1 = B_1; \quad C_2 = \frac{C_1\mathrm{Cos}\sqrt{\phi_2}}{\mathrm{Sin}\sqrt{\phi_2}}$$

$$C_3 = \frac{-\left(D_5 mhM\left(1+e^{-2M}\right) + D_6\left(1-e^{-2M}\right)\right)}{mhM\left(1+e^{-2M}\right)\left(1-e^{-2\sqrt{A_3}}\right) + \sqrt{A_3}\left(1-e^{-2M}\right)\left(1+e^{-2\sqrt{A_3}}\right)}$$

$$C_4 = \frac{D_3 - C_3 e^{-\sqrt{A_3}}}{e^{\sqrt{A_3}}}; \qquad C_5 = B_5 - D_8;$$

$$C_6 = \frac{C_5\mathrm{Cos}\sqrt{\phi_2} - D_9}{\mathrm{Sin}\sqrt{\phi_2}}; \quad D_1 = -\left(l_1\mathrm{Cos}\sqrt{\phi_1} + l_2\mathrm{Sin}\sqrt{\phi_1} + l_3\right);$$

$$D_2 = l_4 + l_6 - l_1 - l_3; \quad D_3 = -\left(l_4\mathrm{Cos}\sqrt{\phi_2} - l_5\mathrm{Sin}\sqrt{\phi_2} + l_6\right)$$

$$D_4 = l_5\sqrt{\phi_2} - mhl_2\sqrt{\phi_1}; \quad D_5 = D_2 - D_1 e^{-M} + D_3 e^{-\sqrt{A_3}}$$

$$D_6 = D_4 - mhMD_1 e^{-M} - D_3\sqrt{A_3}e^{-\sqrt{A_3}}$$

$$D_7 = -\begin{pmatrix} r_{22}e^{2M} + r_{23}e^{-2M} + r_{14}\mathrm{Cos}2\sqrt{\phi_1} + r_{15}\mathrm{Sin}2\sqrt{\phi_1} + \\ r_{16}e^{M}\mathrm{Cos}\sqrt{\phi_1} + r_{18}e^{-M}\mathrm{Cos}\sqrt{\phi_1} + r_{17}e^{M}\mathrm{Sin}\sqrt{\phi_1} \\ + r_{19}e^{-M}\mathrm{Sin}\sqrt{\phi_1} + r_{21}\mathrm{Sin}\sqrt{\phi_1} + r_{20}\mathrm{Cos}\sqrt{\phi_1} + \\ r_{24}e^{M} + r_{25}e^{-M} + r_{26} \end{pmatrix}$$

$$D_8 = \begin{pmatrix} g_{22} + g_{23} + g_{24} + g_{16} + g_{18} + g_{24} + g_{25} + g_{26} \\ -r_{22} - r_{23} - r_{14} - r_{16} - r_{18} - r_{24} - r_{25} - r_{26} \end{pmatrix}$$

$$D_9 = -\begin{pmatrix} g_{22}e^{-2\sqrt{A_3}} + g_{23}e^{2\sqrt{A_3}} + g_{16}\mathrm{Cos}2\sqrt{\phi_2} - g_{15}\mathrm{Sin}2\sqrt{\phi_2} \\ + g_{16}e^{-\sqrt{A_3}}\mathrm{Cos}\sqrt{\phi_2} + g_{18}e^{\sqrt{A_3}}\mathrm{Cos}\sqrt{\phi_2} \\ - g_{17}e^{-\sqrt{A_3}}\mathrm{Sin}\sqrt{\phi_2} - g_{19}e^{\sqrt{A_3}}\mathrm{Sin}\sqrt{\phi_2} + g_{21}\mathrm{Sin}\sqrt{\phi_2} \\ - g_{20}\mathrm{Cos}\sqrt{\phi_2} + g_{24}e^{-\sqrt{A_3}} + g_{25}e^{\sqrt{A_3}} + g_{26} \end{pmatrix}$$

$$D_{10} = 2\sqrt{A_3}\,g_{22} - 2\sqrt{A_3}g_{23} + 2\sqrt{\phi_2}g_{15} + \sqrt{A_3}g_{16} - \sqrt{A_3}g_{18} + \sqrt{\phi_2}g_{17} + \sqrt{\phi_2}g_{19} + g_{20} + \sqrt{A_3}g - \sqrt{A_3}g_{25} - 2Mkhr_{22} + 2Mkhr_{23} - 2kh\sqrt{\phi_1}r_{15} - Mkhr_{16} + Mkhr_{18} - kh\sqrt{\phi_1}r_{17} - kh\sqrt{\phi_1}r_{19} - khr_{20} - Mkhr_{24} + Mkhr_{25}$$

$$D_{11} = l_7\mathrm{Cos}2\sqrt{\phi_1} + l_8\mathrm{Sin}2\sqrt{\phi_1} + l_9e^M\mathrm{Cos}\sqrt{\phi_1} + l_{10}e^M\mathrm{Sin}\sqrt{\phi_1} + l_{11}e^{-M}\mathrm{Cos}\sqrt{\phi_1} + l_{12}e^{-M}\mathrm{Sin}\sqrt{\phi_1} + l_{13}\mathrm{Cos}\sqrt{\phi_1} + l_{14}\mathrm{Sin}\sqrt{\phi_1} + l_{15}\mathrm{Cos}\sqrt{\phi_1} + l_{16}\mathrm{Sin}\sqrt{\phi_1} + l_{17}e^{2M} + l_{18}e^{-2M} + l_{19}e^M + l_{20}e^{-M} + l_{21}$$

$$D_{12} = l_{30} + l_{22} + l_{24} + l_{26} + l_{32} + l_{33} + l_{36} - l_{15} - l_7 - l_9 - l_{11} - l_{17} - l_{18} + l_{19} + l_{20} + l_{21}$$

$$g_1 = C_3^2(A_6A_3 + A_7); \; g_2 = C_4^2(A_6A_3 + A_7)$$

$$g_3 = \frac{A_6\phi_2 l_5^2 - A_6\phi_2 l_4^2 + A_7 l_4^2 - A_7 l_5^2}{2};$$

$$g_4 = A_7 l_4 l_5 - A_6 l_4 l_5 \phi_2$$

$$g_5 = 2\left(A_7C_3l_5 - A_6C_3l_4\sqrt{A_3}\sqrt{\phi_2}\right);$$

$$g_6 = 2\left(A_7C_4l_5 + A_6C_4l_4\sqrt{A_3}\sqrt{\phi_2}\right);$$

$$g_7 = 2\left(A_7C_3l_4 + A_6C_3l_5\sqrt{A_3}\sqrt{\phi_2}\right);$$

$$g_8 = 2\left(A_7C_4l_4 - A_6C_4l_5\sqrt{A_3}\sqrt{\phi_2}\right);$$

$$g_9 = 2(A_7C_3l_6 + A_7EC_3); \; g_{10} = 2(A_7C_4l_6 + A_7EC_4)$$

$$g_{11} = 2(A_7l_4l_6 + A_7El_4); \; g_{12} = 2(A_7l_5l_6 + A_7El_5)$$

$$g_{13} = \frac{\left(l_4^2 + l_5^2\right)\left(A_6\phi_2 + A_7\right) + 2A_7\left(E^2 + l_6^2\right) + 4A_7El_6}{2} - \frac{4A_7El_6 - 4A_6A_3C_3C_4 + 4A_7C_3C_4}{2})$$

$$g_{14} = \frac{g_3}{3\phi_2}; \; g_{15} = \frac{g_4}{3\phi_2}; \quad g_{16} = \frac{2\sqrt{A_3}\sqrt{\phi_2}g_5 - g_7A_3}{A_3(A_3 + 4\phi_2)}$$

$$g_{17} = \frac{-\left(2\sqrt{A_3}\sqrt{\phi_2}g_7 + g_5A_3\right)}{A_3(A_3 + 4\phi_2)};$$

$$g_{18} = \frac{-\left(2\sqrt{A_3}\sqrt{\phi_2}g_6 + g_8A_3\right)}{A_3(A_3 + 4\phi_2)}$$

$$g_{19} = \frac{\left(2\sqrt{A_3}\sqrt{\phi_2}g_8 - g_6A_3\right)}{A_3(A_3 + 4\phi_2)}; \quad g_{20} = \frac{g_{12}}{2\sqrt{\phi_2}};$$

$$g_{21} = \frac{-g_{11}}{2\sqrt{\phi_2}}; \; g_{22} = \frac{-g_1}{4A_3 + \phi_2}; \; g_{23} = \frac{-g_2}{4A_3 + \phi_2};$$

$$g_{24} = \frac{-g_9}{(A_3 + \phi_2)}; g_{25} = \frac{-g_{10}}{(A_3 + \phi_2)}; \; g_{26} = \frac{-g_{13}}{\phi_2};$$

$$l_1 = \frac{A_1B_1}{M^2 + \phi_1}; \quad l_2 = \frac{A_1B_2}{M^2 + \phi_1}; \quad l_3 = \frac{-A_2}{M^2}$$

$$l_4=\frac{A_5C_1}{A_3+\phi_2};\ l_5=\frac{A_5C_2}{A_3+\phi_2};\ l_6=\frac{-A_4}{A_3}$$

$$l_7=\frac{A_1r_{14}}{\left(M^2+4\phi_1\right)};\quad l_8=\frac{A_1r_{15}}{\left(M^2+4\phi_1\right)}$$

$$l_9=\frac{A_1r_{16}\phi_1+2A_1M\sqrt{\phi_1}r_{17}}{\phi_1\left(4M^2+4\phi_1\right)};$$

$$l_{10}=\frac{A_1r_{17}\phi_1-2A_1M\sqrt{\phi_1}r_{16}}{\phi_1\left(4M^2+4\phi_1\right)};$$

$$l_{11}=\frac{A_1r_{18}\phi_1-2A_1M\sqrt{\phi_1}r_{19}}{\phi_1\left(4M^2+\phi_1\right)};$$

$$l_{12}\frac{A_1r_{19}\phi_1+2A_1M\sqrt{\phi_1}r_{18}}{\phi_1\left(4M^2+\phi_1\right)};\ l_{13}=\frac{A_1r_{20}}{M^2+\phi_1};$$

$$l_{14}=\frac{A_1r_{21}}{M^2+\phi_1};\ l_{15}=\frac{A_1B_5\left(M^2+\phi_1\right)+2A_1\sqrt{\phi_1}r_{21}}{\left(M^2+\phi_1\right)^2};$$

$$l_{16}=\frac{A_1B_6\left(M^2+\phi_1\right)-2A_1\sqrt{\phi_1}r_{20}}{\left(M^2+\phi_1\right)^2}$$

$$l_{17}=\frac{-A_1r_{22}}{3M^2};\quad l_{18}=\frac{-A_1r_{23}}{3M^2};\quad l_{19}=\frac{-A_1r_{24}}{2M}$$

$$l_{20}=\frac{A_1r_{25}}{2M};\ l_{21}=\frac{A_1r_{26}}{M^2};\ l_{22}=\frac{A_5g_{14}}{4\phi_2+A_3};$$

$$l_{23}=\frac{A_5g_{17}}{4\phi_2+A_3};\ l_{24}=\frac{A_5g_{16}\phi_2+2A_5\sqrt{A_3}\sqrt{\phi_2}g_{17}}{\phi_2(4A_3+\phi_2)};$$

$$l_{25}=\frac{A_5g_{17}\phi_2-2A_5\sqrt{A_3}\sqrt{\phi_2}g_{16}}{A_3(4A_3+\phi_2)};$$

$$l_{26}=\frac{A_5g_{18}\phi_2-2A_5\sqrt{A_3}\sqrt{\phi_2}g_{19}}{\phi_2(4A_3+\phi_2)};$$

$$l_{27}=\frac{A_5g_{19}\phi_2+2A_5\sqrt{A_3}\sqrt{\phi_2}g_{18}}{\phi_2(4A_3+\phi_2)}$$

$$l_{28}=\frac{A_5g_{20}}{\phi_2+A_3};\quad l_{29}=\frac{A_5g_{21}}{\phi_2+A_3};$$

$$l_{30}=\frac{A_5C_5(\phi_2+A_3)+2A_5\sqrt{\phi_2}g_{21}}{(\phi_2+A_3)^2}$$

$$l_{31}=\frac{A_5C_6(\phi_2+A_3)-2A_5\sqrt{\phi_2}g_{20}}{(\phi_2+A_3)^2};\quad l_{32}=\frac{-A_5g_{22}}{3A_3};$$

$$l_{33}=\frac{-A_5g_{23}}{3A_3}\quad l_{34}=\frac{-A_5g_{24}}{2\sqrt{A_3}};\quad l_{35}=\frac{A_5g_{25}}{2\sqrt{A_3}};$$

$$l_{36}=\frac{A_5g_{26}}{A_3};\quad r_1=2B_3^2M^2;\qquad r_2=2B_4^2M^2;$$

$$r_3=\frac{l_2^2\phi_1+M^2l_1^2-l_1^2\phi_1-l_2^2M^2}{2}$$

$$r_4=M^2l_1l_2-l_1l_2\phi_1;\ r_5=2\left(B_3Ml_2\sqrt{\phi_1}+M^2B_3l_1\right);$$

$$r_6=2\left(M^2B_4l_1-B_4Ml_2\sqrt{\phi_1}\right);$$

$$r_7=2\left(M^2B_3l_2-B_3Ml_1\sqrt{\phi_1}\right);$$

$$r_8=2\left(B_4Ml_1\sqrt{\phi_1}+M^2B_4l_2\right);$$

$$r_9=2M^2l_1\left(l_3+E\right);\qquad r_{10}=2M^2l_2\left(l_3+E\right)$$

$$r_{11}=2M^2B_3\left(l_3+E\right);\qquad r_{12}=2M^2B_4\left(l_3+E\right);$$

$$r_{13}=\frac{2M^2\left(E+l_3\right)^2+\left(l_2^2+l_1^2\right)\left(M^2+\phi_1\right)}{2};$$

$$r_{14}=\frac{r_3}{3\phi_1};\qquad r_{15}=\frac{r_4}{3\phi_1};$$

$$r_{16}=\left(\frac{2\sqrt{\phi_1}r_7}{M^3+4M\phi_1}-\frac{r_5}{M^2+4\phi_1}\right);$$

$$r_{17}=-\left(\frac{2\sqrt{\phi_1}r_5}{M^3+4M\phi_1}+\frac{r_7}{M^2+4\phi_1}\right);$$

$$r_{18}=-\left(\frac{2\sqrt{\phi_1}r_8}{M^3+4M\phi_1}+\frac{r_6}{M^2+4\phi_1}\right);$$

$$r_{19}=\left(\frac{2r_6\sqrt{\phi_1}}{M^3+4M\phi_1}-\frac{r_8}{M^2+4\phi_1}\right);\ r_{20}=\frac{r_{10}}{2\sqrt{\phi_1}};$$

$$r_{21}=\frac{-r_9}{2\sqrt{\phi_1}};\ r_{22}=\frac{-r_1}{4M^2+\phi_1};\ r_{23}=\frac{-r_2}{4M^2+\phi_1};$$

$$r_{24}=\frac{-r_{11}}{M^2+\phi_1};\ r_{25}=\frac{-r_{12}}{M^2+\phi_1};\ r_{26}=\frac{-r_{13}}{\phi_1}$$