OSCILLATORY FLOW AND HEAT TRANSFER IN A HORIZONTAL COMPOSITE POROUS MEDIUM CHANNEL

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ABSTRACT

A study of time-dependent oscillatory fluid flow and heat transfer in a horizontal composite porous medium channel has been carried out. The channel walls are maintained at two different constant temperatures. The partial differential equations governing the flow and heat transfer are transformed to ordinary differential equations by making certain physical assumptions and closed-form solutions are obtained. Separate solutions for the porous medium and viscous fluid regions are obtained and these solutions are matched at the interface using suitable matching conditions. The closed-form solutions are evaluated numerically and the results are presented graphically for various values of flow governing parameters such as the permeability parameter, viscosity ratio, oscillation amplitude, conductivity ratio, Prandtl number and the Eckert number. The effect of periodic frequency on the flow is also depicted in tabular form.

KEYWORDS: Oscillatory flow, porous medium channel, fluid suction, heat transfer, analytical solutions

1. NOMENCLATURE

A real positive constant
Cp specific heat at constant pressure
Ec Eckert number
h channel half width
g gravitational acceleration
K thermal conductivity
Keff effective thermal conductivity
k ratio of thermal conductivity
m ratio of viscosity
P non-dimensional pressure gradient
p pressure
Pr Prandtl number
s permeability of porous matrix
T temperature
Tw wall temperature
t time
u,v velocity components of velocity along and perpendicular to the plates, respectively
U0 average velocity
v0 scale of suction
x,y cartesian coordinates

Greek letters

ε coefficient of periodic parameter
δ Kronecker delta
ρ0 fluid density
μ fluid dynamic viscosity
μeff effective viscosity
ω frequency parameter
ωt periodic frequency parameter
ν kinematic viscosity
σ porous medium parameter

Subscripts

1, 2 quantities for Region-I and Region-II, respectively.

1. INTRODUCTION

The systematic study of flow past a porous medium constitutes a comparatively recent development in fluid mechanics, with applications in science, engineering, and technology. The existence of a fluid layer adjacent to a layer of fluid-saturated porous medium is a common occurrence in both geophysical and engineering environments. Freezing of soils and melting of ice in frozen soils due to the change in weather conditions also require the knowledge of interaction mechanism between the fluid and porous layer. In the presence of salt gradients (coastal areas) and/or polluting materials, the melting or freezing of water-saturated soil may exhibit fingering phenomena. Composite fluid and porous layers also find its application in the porous journal bearings.
The composite systems also exist in numerous other engineering applications such as fibrous and granular insulation where the insulation occupies only part of the space separating the heated and cooled walls, porous insulation of ducts which permit convective interaction between the duct walls and the ambient air heat transfer from hair covered skin, grain storage and drying, drying paper, and food preparation and storage. In all of the above-mentioned composite systems, fluid motion in one layer is not independent of that in the other, the interaction being dictated by the conditions at the interface between the two layers.

Another important application of porous boundaries is in providing effective insulation. In order to obtain the most effective insulation for a given thermal condition, it is desirable to minimize solid conduction and maximize porosity in such a way as to reduce the apparent thermal conductivity [1].

Fundamental investigations in saturated porous media appear to have started with linearized stability theory applied to an infinite horizontal layer heated from below [2,3]. Extensive reviews of prior work have been presented by Combarnous and Bories [4] and Cheng [5]. Rudraiah [6] analyzed flow through porous medium of modified BJR (Beavers-Joseph-Rudraiah) slip condition in the case of finite thickness of the porous layer.


All above investigators have only considered steady flow. However, for many problems of practical interest the flow may be unsteady. Soundalgkar and Bhat [16] analyzed oscillatory MHD channel flow and heat transfer under a transverse magnetic field. Umavathi and Palaniappan [17] studied oscillatory flow of unsteady convective fluid in an infinite vertical porous stratum. Chamkha [18] dealt with unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption. Recently, Umavathi and co-workers [19-21] analyzed oscillatory flow and heat transfer for viscous and electrically conducting immiscible fluids in a horizontal channel.

Keeping in view the wide area of practical importance of flow through composite porous media, it is the objective of the present study to analyze unsteady flow and heat transfer in a horizontal channel bounded below by a porous matrix and above by clear viscous fluid.

3. MODEL EQUATIONS

The complete system of momentum and energy equations are given by

Region-I (porous medium region):

\[ \rho_0 \left( \frac{\partial \bar{q}_1}{\partial t} + \bar{q}_1 \cdot \nabla \bar{q}_1 \right) = -\nabla P - \frac{h l}{s} \bar{q}_1 + \mu_{eff} V^2 \bar{q}_1 \]

(1)

where \( \Phi_1 \) and \( \Phi_2 \) are the viscous dissipation terms and second term in equation (2) is the Darcy dissipation. The conservation equations for porous region are based on a Darcian model incorporating the viscous and Darcy dissipation terms in the energy equation. For moderate values of the velocities and viscosities of the fluids, the dissipation terms are important and therefore included in the energy equations. The physical configuration considered in this study is shown in Figure 1.

![Physical configuration](image)

We consider a two dimensional fluid flow in a horizontal channel when the suction velocity varies periodically with time about a non-zero constant mean \( v_0 \). The region \(-h < y < 0\) (Region-I) is filled with porous matrix and the region \( 0 < y < h \) (Region-II) is occupied by clear fluid. The two walls of the channel are held at constant different temperature \( T_{w_1} \) and \( T_{w_2} \) with \( T_{w_1} > T_{w_2} \) and that plates are placed horizontally. The porous medium is considered to be homogenous and isotropic. The fluid within the porous medium saturates the solid matrix and both are in local thermodynamic equilibrium. The flow in both the regions is assumed to be unsteady, fully developed, laminar and incompressible. The thermo-physical properties of the fluid and effective properties of the porous medium are assumed constant. Further the flow in both regions is assumed
to be driven by a common pressure gradient \(-\partial p / \partial x\) and temperature gradient \(\Delta T = T_{w_1} - T_{w_2}\).

With the assumptions mentioned above, the governing equations of continuity, motion and energy reduces to

\[
\frac{\partial v_i}{\partial y} = 0
\]
\[\rho_0 \left( \frac{\partial u_i}{\partial t} + v_i \frac{\partial u_i}{\partial y} \right) = \chi\mu \frac{\partial u_i}{\partial y}^2 - \frac{\partial p}{\partial x} - \frac{\mu}{s} u_i
\]
\[
\rho_0 CP \left( \frac{\partial T_i}{\partial t} + v_i \frac{\partial T_i}{\partial y} \right) = \chi K \frac{\partial^2 T_i}{\partial y^2} + \left( \frac{\partial u_i}{\partial y} \right)^2 + \chi \frac{\mu}{s} u_i^2
\]

where \(i=1,2\) gives equations for Regions I and II, respectively and

\[
\chi_\mu = \chi_{\text{eff}} \quad \text{for porous region}
\]
\[
\chi_\mu = \chi \quad \text{for clear fluid region}
\]
\[
\chi_K = \chi_{K_{\text{eff}}} \quad \text{for porous region}
\]
\[
\chi_K = K \quad \text{for clear fluid region}
\]
\[
\chi = 1 \quad \text{for porous region}
\]
\[
\chi = 0 \quad \text{for clear fluid region}
\]

Mathematically, the problem involves the coupling of the governing equations for the fluid region with the equations for the porous region through an appropriate set of matching conditions at the fluid porous medium interface. We assume the continuity of velocity, shear stress, temperature and heat flux at the interface. The boundary conditions on velocity are the no-slip boundary conditions, which required that the x-component of velocity must vanish at the wall. The boundary conditions on temperature are isothermal conditions.

The appropriate boundary and matching conditions for the problem under consideration can be written as

\[
\begin{align*}
u_i(-h) &= 0 \\
u_i(h) &= 0 \\
u_i(0) &= u_2(0) \\
\chi_{\text{eff}} \frac{\partial u_i}{\partial y} &= \chi \frac{\partial u_2}{\partial y} & \text{at } y = 0 \\
T_i(-h) &= T_{w1} \\
T_i(h) &= T_{w2} \\
T_i(0) &= T_2(0) \\
\chi_{K_{\text{eff}}} \frac{\partial T_i}{\partial y} &= K \frac{\partial T_2}{\partial y} & \text{at } y = 0
\end{align*}
\]

The continuity equations of both fluids [Equation (5)] imply that, \(v_1\) and \(v_2\) are independent of \(y\), they can be utmost a function of time alone. Hence, we can write (assuming \(v_1 = v_2 = v\))

\[
v = v_0 \left( 1 + \varepsilon A e^{i\omega t} \right)
\]

where \(A\) is a real positive constant, \(\omega\) is frequency parameter and \(\varepsilon\) is small such that \(\varepsilon A \leq 1\). Here, it is assumed that the transpiration velocity varies periodically with time about a non-zero constant mean \(v_0\). When \(\varepsilon A = 0\), the case of constant transpiration velocity is recovered. By use of the following non-dimensional quantities

\[
\begin{align*}
u_i &= U_0 u_i^* \quad y = \frac{y}{v_0} \quad t = \frac{v_0 T_i}{v_0} \quad \nu = v_0 v_i^* \quad \sigma^2 = \frac{v_0^2}{S v_0^2} \\
P &= \frac{v_0^2}{\mu v_0^2 U_0} \left( -\frac{\partial P}{\partial x} \right) \quad \theta = \frac{T - T_{w2}}{T_{w1} - T_{w2}} \quad Ec = \frac{U_0^2}{C_p \Delta T}
\end{align*}
\]

Equations (6) and (7) are placed in dimensionless form as (dropping asterisks),

\[
\frac{\partial u_1}{\partial t} + \nu \frac{\partial u_1}{\partial y} = \frac{A_1}{\nu} \frac{\partial^2 u_1}{\partial y^2} - \chi \sigma^2 u_1 + P
\]
\[
\frac{\partial \theta_1}{\partial t} + \nu \frac{\partial \theta_1}{\partial y} = \frac{B_1}{\nu} \frac{\partial^2 \theta_1}{\partial y^2} + A_1 \left( \frac{\partial u_1}{\partial y} \right)^2 + \chi \sigma^2 Ec u_1
\]

where \(i=1,2\) give equations for Regions I and II and

\[
\begin{align*}
A_1 &= m; \quad A_2 = I; \quad B_1 = k / \nu; \quad B_2 = 1 / \nu; \quad Pr = \rho v_0 C_p / K; \quad \text{and}
\end{align*}
\]

\[
\chi = \mu_{\text{eff}} / \mu \quad \text{is the ratio of viscosities and } k = K_{\text{eff}} / K \quad \text{is the ratio of thermal conductivities.}
\]

The hydrodynamic and thermal boundary and interface conditions for both regions in non-dimensional form become

\[
\begin{align*}
u_i(-1) &= 0 \\
u_i(1) &= 0 \\
u_i(0) &= u_2(0) \\
\nu \chi_{\text{eff}} \frac{\partial u_1}{\partial y} &= \nu \chi \frac{\partial u_2}{\partial y} & \text{at } y = 0 \\
\theta_i(-1) &= 1 \\
\theta_i(1) &= 0 \\
\theta_i(0) &= \theta_2(0) \\
\nu k \frac{\partial \theta_1}{\partial y} &= \nu k \frac{\partial \theta_2}{\partial y} & \text{at } y = 0
\end{align*}
\]

4. CLOSED-FORM SOLUTIONS

Equations (12) and (13) are coupled partial differential equations and hence finding exact solutions is out of scope. However, they can be reduced to a set of ordinary differential equations that can be solved analytically. This is done by assuming

\[
\begin{align*}
u_i(y, t) &= u_{i0}(y) + e^{i\omega t} u_{i1}(y) + O(\varepsilon^2) + ... \\
\theta_i(y, t) &= \theta_{i0}(y) + e^{i\omega t} \theta_{i1}(y) + O(\varepsilon^2) + ...
\end{align*}
\]

for \(i = 1, 2\)

This is a valid assumption because of the choice of \(\nu\) as defined in Equation (10) that the amplitude \(\varepsilon A \ll 1\).
Considering the real part of $e^{ist}$ equations (16) and (17) becomes

$$u_i(y,t) = u_{i0}(y) + \varepsilon \cos \omega t \ u_{i1}(y) + ...$$  \hspace{1cm} (18)

$$\theta_i(y,t) = \theta_{i0}(y) + \varepsilon \cos \omega t \ \theta_{i1}(y) + ...$$  \hspace{1cm} (19)

for $i = 1,2$

By substituting equations (18) and (19) into equations (12) and (13), equating the harmonic and non-harmonic terms and neglecting the higher order terms of $O(\varepsilon^2)$, one obtains the following pairs of equations for $(u_{i0}, \theta_{i0})$ and $(u_{i1}, \theta_{i1})$.

**Non-periodic coefficients $O(\varepsilon^0)$**

$$A_1 \frac{d^2u_{i0}}{dy^2} - \frac{du_{i0}}{dy} + \chi \sigma^2 u_{i0} = -P$$  \hspace{1cm} (20)

$$B_1 \frac{d^2\theta_{i0}}{dy^2} - \frac{d\theta_{i0}}{dy} = -A_1 \varepsilon \left( \frac{du_{i0}}{dy} \right)^2 - \chi \sigma^2 E \varepsilon u_{i0}^2$$  \hspace{1cm} (21)

for $i = 1,2$

**Periodic coefficients $O(\varepsilon^1)$**

$$A_1 \frac{d^2u_{i1}}{dy^2} + \left( \omega \tan \omega t - \chi \sigma^2 \right) u_{i1} = A \frac{du_{i0}}{dy}$$  \hspace{1cm} (22)

$$B_1 \frac{d^2\theta_{i1}}{dy^2} + \omega \tan \omega t \ \theta_{i1} = A \frac{du_{i0}}{dy}$$  \hspace{1cm} (23)

$$-2E \varepsilon A_1 \frac{du_{i0}}{dy} \frac{du_{i1}}{dy} - 2\chi \sigma^2 E \varepsilon u_{i0} u_{i1}$$ for $i = 1,2$

The corresponding boundary and interface conditions can be written as

$$u_i(-1) = 0$$
$$u_i(0) = u_{i2}(0)$$
$$u_{i1}(0) = 0$$

$$m \frac{\partial u_{i1}}{\partial y} = \frac{\partial u_{i1}}{\partial y} \text{ at } y = 0$$
$$\theta_i(-1) = 1 - \delta_{il}$$
$$\theta_i(0) = \theta_{i1}(0)$$
$$\theta_{i1}(1) = 0$$

$$k \frac{\partial \theta_{i1}}{\partial y} = \frac{\partial \theta_{i1}}{\partial y} \text{ at } y = 0$$

where $i=0,1$ give the boundary and interface conditions for non-periodic ($O(\varepsilon^0)$) and periodic ($O(\varepsilon^1)$) coefficients respectively and $\delta_{il}$ is the Kronecker delta.

Solutions of non-periodic (harmonic) terms lead to steady flow solutions which are given by

$$u_{20} = C_3 + C_4 e^{\gamma y} + P_y$$  \hspace{1cm} (27)

$$\theta_{10} = C_5 + C_6 e^{\alpha y} + k_8 y + k_{10} e^{\alpha y} + k_{11} e^{\alpha y} + k_{12} e^{2\alpha y} + k_{13} e^{2\alpha y} + k_{14} e^{4\alpha y}$$  \hspace{1cm} (28)

$$\theta_{20} = C_7 + C_8 e^{\beta y} + k_{15} e^{2y} + k_{16} e^{2y} + k_{17} y$$  \hspace{1cm} (29)

Solutions of periodic (non-harmonic) terms in both regions of the channel take on different forms depending on the value of $4\omega \tan \omega t$ for both velocity and temperature. These forms can be shown as

**Case-I:**

**Region-I**

$$u_{11} = C_4 e^{\alpha y} + C_{10} e^{\alpha y} + k_{18} e^{\alpha y} + k_{19} e^{\alpha y} + k_{20} e^{2\alpha y} + k_{21} e^{2\alpha y}$$  \hspace{1cm} (30)

for $4m(\omega \tan \omega t - \sigma^2) < 1$

$$\theta_{11} = C_{23} + C_{24} e^{\alpha y} + k_{25} e^{\alpha y} + k_{26} e^{2\alpha y} + k_{27} e^{2\alpha y} + k_{28} e^{2\alpha y} + k_{29} e^{4\alpha y} + k_{30} e^{4\alpha y}$$  \hspace{1cm} (31)

$$+ k_{31} e^{6\alpha y} + k_{32} e^{6\alpha y} + k_{33} e^{6\alpha y} + k_{34} e^{6\alpha y} + k_{35} e^{6\alpha y} + k_{36} e^{6\alpha y} + k_{37} e^{6\alpha y} + k_{38} e^{6\alpha y}$$

for $4\omega \tan \omega t < Pr$

**Region-II**

$$u_{21} = C_4 e^{\alpha y} + C_{12} e^{\alpha y} + k_{21} e^{\alpha y} + k_{22} e^{\alpha y}$$  \hspace{1cm} (32)

for $4\omega \tan \omega t < 1$

$$\theta_{21} = C_{23} + C_{24} e^{\alpha y} + C_{16} e^{\alpha y} + k_{25} e^{\alpha y} + k_{26} e^{\alpha y} + k_{27} e^{\alpha y} + k_{28} e^{\alpha y}$$

$$+ k_{29} e^{2\alpha y} + k_{30} e^{2\alpha y} + k_{31} e^{2\alpha y} + k_{32} e^{2\alpha y} + k_{33} e^{2\alpha y} + k_{34} e^{2\alpha y}$$

$$+ k_{35} e^{2\alpha y} + k_{36} e^{2\alpha y} + k_{37} e^{2\alpha y} + k_{38} e^{2\alpha y}$$

for $4\omega \tan \omega t < Pr$

**Case-II:**

**Region-I**

$$u_{11} = (D_1 + D_2 y) e^{\alpha y} + k_{18} e^{\alpha y} + k_{19} e^{\alpha y}$$  \hspace{1cm} (34)

for $4m(\omega \tan \omega t - \sigma^2) = 1$

$$\theta_{11} = (D_3 + D_4 y) e^{\alpha y} + P_{27} e^{2\alpha y} + P_{38} e^{2\alpha y} + P_{29} e^{\alpha y}$$

$$+ P_{30} e^{\alpha y} + P_{31} e^{\alpha y} + k_{21} e^{2\alpha y} + k_{22} e^{2\alpha y} + k_{23} e^{2\alpha y}$$

$$+ k_{24} e^{2\alpha y} + k_{25} e^{2\alpha y} + k_{26} e^{2\alpha y} + k_{27} e^{2\alpha y} + k_{28} e^{2\alpha y}$$

for $4\omega \tan \omega t = Pr$

**Region-II**

$$u_{21} = (D_3 + D_4 y) e^{\alpha y} + k_{21} e^{\alpha y} + k_{22} e^{\alpha y}$$  \hspace{1cm} (36)

for $4\omega \tan \omega t = 1$

$$\theta_{21} = (D_3 + D_8 y) e^{\alpha y} + P_{38} e^{\alpha y} + K_{57} e^{\alpha y} + k_{56} e^{\alpha y}$$

$$+ P_{36} e^{\alpha y} + P_{37} e^{\alpha y} + k_{21} e^{2\alpha y} + k_{22} e^{2\alpha y} + k_{23} e^{2\alpha y} + k_{24} e^{2\alpha y}$$

for $4\omega \tan \omega t = Pr$

**Case-III:**

**Region-I**

$$u_{11} = e^{\alpha y} (e^{\alpha y} e^{\alpha y} + e^{\alpha y} e^{\alpha y} + k_{18} e^{\alpha y} + k_{19} e^{\alpha y}$$  \hspace{1cm} (38)

for $4m(\omega \tan \omega t - \sigma^2) > 1$
\[ \theta_{11} = e^{a_1 y} (e_1 \cos \beta_2 y + e_4 \sin \beta_3 y) \\
+ e^{a_2 y} (R_{2e} \cos \beta_1 y - R_{3e} \sin \beta_2 y) \\
+ e^{a_3 y} (R_{3e} \cos \beta_1 y - R_{3e} \sin \beta_1 y) \\
+ R_{2e} e^{a_2 y} (R_{2e} \cos \beta_1 y - R_{3e} \sin \beta_2 y) + R_{3e} e^{a_3 y} + R_{3e} e^{a_3 y} + K_{2e} e^{a_3 y} + K_{24} \] (39)

for \( 4 \cot \alpha > \Pr \)

\[ \theta_{21} = e^{a_1 y} (e_1 \cos \beta_2 y + e_4 \sin \beta_2 y) + k_{21} e^y + k_{22} \] (40)

for \( 4 \cot \alpha > 1 \)

\[ \theta_{21} = e^{a_1 y} (e_1 \cos \beta_2 y + e_4 \sin \beta_2 y) + R_{4e} e^{a_3 y} (R_{3e} \cos \beta_2 y - R_{4e} \sin \beta_2 y) \\
+ R_{4e} e^{a_3 y} (R_{4e} \cos \beta_2 y - R_{4e} \sin \beta_2 y) \\
+ R_{4e} e^{a_3 y} + R_{4e} e^{a_3 y} + K_{5e} e^{a_3 y} + K_{60} \] (41)

for \( 4 \cot \alpha > \Pr \)

It should be noted that all of the constants appearing in the above solutions are defined in the Appendix. The value of \( \varepsilon \) is fixed as 0.01, A as 0.1, P as 5, m as 1, k as 1 except the varying one.

5. RESULTS AND DISCUSSION

Figures 2 and 3 present typical velocity and temperature profiles for different values of the porous parameter \( \sigma \). For large values of the porous parameter \( \sigma \), the frictional drag is very large and as a result, the velocity is very small in the porous medium region of the channel. As the porous parameter \( \sigma \) decreases, both the velocity and temperature increase in the porous medium region as well as in the clear viscous fluid region of the channel. The magnitude of increase is large in the clear viscous fluid region compared to the porous medium region.

Figures 4 and 5 depict the effect of the viscosity ratio on the velocity and temperature profiles in the channel, respectively. As the viscosity ratio increases, both the velocity and temperature profiles are suppressed. This is due to the relation between the frequency parameter and the tangential periodic frequency parameter \( 4m(\omega \tan \alpha - \sigma^2) < 1 \). Physically, this relation implies that the fluid in the lower region becomes thick and hence, the flow is suppressed. The magnitude of suppression is larger in the porous medium region than that in the clear viscous fluid region of the channel.
Figures 6 and 7 illustrate the effect of the oscillation amplitude on the velocity and temperature fields, respectively. As the oscillation amplitude increases, the fluid velocity increases but its temperature decreases. The effect of the conductivity ratio \( k \) on the temperature profile is shown in Fig. 8. As the conductivity ratio increases, the temperature decreases in the porous medium region whereas it increases in the clear viscous fluid region. That is, the conductivity ratio has the tendency to increase the thermal state in the clear viscous fluid region and to reduce it in the porous medium region of the channel. As the Prandtl number and the Eckert number increase, the fluid temperature increases as seen in Figs. 9 and 10, respectively. This is due to the fact that the Prandtl number is a measure of the viscous and heat conduction effects and the Eckert number represents the effects of the viscous and porous medium dissipations. It should be noted that Figs. 2, 4 and 6 are drawn for \( 4m(\omega \tan \omega t - \omega^2) < 1 \) and Figs. 3, 5, 7, 8, 9 and 10 for \( 4k \omega \tan \omega t < Pr \).
Considerably in the porous medium region when compared to the clear viscous fluid region. It should be noted that Figs. 11 and 13 are drawn for $4m(o\tan \omega t - \sigma^2) = 1$ and Figs. 12 and 14 are drawn for $4k\tan \omega t = Pr$.

The effect of the porous parameter, viscosity ratio and the frequency parameter on the velocity profiles for $4m(o\tan \omega t - \sigma^2) < 1$ and the temperature profiles for $4k\tan \omega t < Pr$ and the effect of the conductivity ratio, Prandtl number and Eckert number on the temperature profiles for $4k\tan \omega t < Pr$, show the same nature as that for $4m(o\tan \omega t - \sigma^2) > 1$ and $4k\tan \omega t > Pr$ and are not presented here for lack of space.

Tables 1-3 display the effect of the periodic frequency parameter $o\omega t$ on the flow field. Table 1 shows this effect when $4m(o\tan \omega t - \sigma^2) < 1$ for velocity and $4k\tan \omega t < Pr$ for temperature, Table 2 shows this effect for the case when $4m(o\tan \omega t - \sigma^2) = 1$ for velocity and $4k\tan \omega t = Pr$ for temperature and Table 3 presents this effect for the case when $4m(o\tan \omega t - \sigma^2) > 1$ for velocity and $4k\tan \omega t > Pr$ for temperature, respectively. It is seen from Table 1 that as $o\omega t$ increases the velocity increases.
above $y=0.4$ and decreases below $y=0.4$, whereas the temperature for $\alpha t = 2.3571$ and $\alpha t = 2.6190$ decreases on both sides of the interface when compared to $\alpha t = 2.0952$. The variation on the velocity and temperature is after three and two decimal places, respectively. As $\alpha t$ increases for the case $4m(\alpha t \tan \alpha t - \sigma^2) = 1$, the velocity decreases above and $y=0.4$ and increases below $y=0.4$ as seen in Table 2. The temperature decreases above $y=0.6$ and increases below $y=0.6$ as $\alpha t$ increases. Here, also the variation on the velocity and temperature values is after three and two decimal places, respectively. Table 3 shows that the velocity decreases above the interface and increases below the interface for $\alpha t = 0.5238$ when compared to $\alpha t = 0.2619$. For $\alpha t = 0.6984$, the velocity increases above the interface and decreases below the interface when compared to $\alpha t = 0.2619$. The temperature increases above $y=0.4$ for $\alpha t = 0.5238$ and 0.6984 when compared to $\alpha t = 0.2619$. The variation is after three decimal places on temperature.

Table 1: Velocity and temperature profiles for different values of the periodic frequency parameter for $\omega = 0.75$, $\varepsilon A = 0.001$ and $Pr = 0.7$.

<table>
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<tr>
<th>$Y$</th>
<th>Velocity</th>
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<tr>
<td></td>
<td>$\alpha t = 2.0952$</td>
<td>$\alpha t = 2.3571$</td>
<td>$\alpha t = 2.6190$</td>
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<td>0.00000</td>
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6. CONCLUSIONS

The problem of oscillatory fluid flow and heat transfer in a horizontal composite porous medium channel was considered. The channel walls were maintained at two different constant temperatures. The governing equations were formulated and then reduced into a set of ordinary differential equations by making certain physical assumptions. Closed-form solutions were obtained for the reduced equations. Separate solutions for the porous medium and clear viscous fluid regions are reported and these solutions were matched at the interface using suitable matching conditions. The closed-form solutions were evaluated numerically and the results for the velocity and temperature fields were presented graphically for various values of parameters such as the permeability parameter, viscosity ratio, oscillation amplitude, periodic frequency, conductivity ratio, Prandtl number and the Eckert number. It was found that significant changes in the velocity and temperature fields in the composite channel occur as a result of altering most of the physical parameters. Thus, one can conclude that the flow can be controlled by altering the porous material, considering different fluids having different values for Prandtl numbers, and altering the thickness of porous matrix which leads to different ratios of physical parameters such as viscosity and conductivity ratios.

Table 2: Velocity and temperature profiles for different values of the frequency and the periodic frequency parameter for $\sigma = 0.4$, $\varepsilon A = 0.001$ and $Pr = 1.0$.

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Table 3: Velocity and temperature profiles for different values of the periodic frequency parameter for \( \omega = 60.0, \varepsilon A = 0.001 \) and \( Pr = 0.7 \).

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<th>( \omega t = 0.5238 )</th>
<th>( \omega t = 0.6984 )</th>
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REFERENCES


Appendix

\[
C_1 = \left( \frac{l_2 e^{-m_2} - l_2 P}{l_2 e^{-m_1} - l_2 e^{-m_2}} \right) \; ; \; C_2 = -e^{-m_2} \left( \frac{P}{\sigma^2} + C_1 e^{-m_1} \right) \; ; \\
C_3 = C_1 + C_2 - C_4 + \frac{P}{\sigma^2} \; ; \; C_4 = m(m_1 C_1 - m_2 C_2) - P \; ; \\
C_5 = -C_6 e^{-m_4} - l_4 \; ; \; C_6 = \frac{l_9 - l_4}{e^{-m_4} - l_8} \; ; \\
C_7 = C_5 + C_6 - C_8 + l_6 \; ; \; C_8 = \left( \frac{k m C_6 + l_7}{Pr} \right) \; ; \\
C_9 = \frac{C_2 e^{-m_6} - l_1 C_10}{l_1 e^{-m_6} - l_1 e^{-m_5}} \; ; \; C_{10} = \frac{C_9 e^{-m_6} + l_10}{e^{-m_6}} \; ; \\
C_{11} = \frac{C_9 + C_{10} + l_{15}}{l_{14}} \; ; \; C_{12} = C_9 + C_{10} - C_{11} + l_{12} \; .
\]
$K_{59} = \frac{PrAK_{16}}{1-Pr+PrK_{20}}$; $K_{60} = AK_{17}/K_{20}$

$K_{61} = \frac{-2E \alpha \rho_{m} C_{e} \gamma_{1}}{m_{1}^{2} Pr \rho_{m} + Pr K_{20}}$; $K_{62} = \frac{-2E \alpha \rho_{m} C_{e} \gamma_{12}}{m_{1}^{2} - Pr \rho_{m_{19}} + Pr K_{20}}$

$K_{63} = \frac{-2E \alpha \rho_{m} C_{e} \gamma_{21}}{4 - 2Pr + Pr K_{20}}$; $K_{64} = \frac{-2E \alpha \rho_{m} C_{e} \gamma_{11}}{m_{1}^{2} - Pr \rho_{m_{20}} + Pr K_{20}}$

$K_{65} = \frac{-2E \alpha \rho_{m} C_{e} \gamma_{11}}{m_{2}^{2} - Pr \rho_{m_{20}} + Pr K_{20}}$; $K_{66} = \frac{-2E \alpha \rho_{m} C_{e} \gamma_{11}}{1 - Pr + Pr K_{20}}$

$K_{67} = K_{58} + K_{63}$; $K_{68} = K_{59} + K_{66}$; $l_{1} = 1 + mm_{1}(e^{l} - 1)$

$P = \frac{B}{\sigma^{2}} - P(e^{l} - 1) + P$

$K_{14} = e^{-m_{3}} - 1$

$n_{1} = \frac{1}{2m_{n}}$; $n_{2} = \frac{1}{2}$; $n_{3} = \frac{1}{2K_{2}}$; $n_{4} = m_{1} + n_{1}$; $n_{5} = m_{2} + n_{1}$

$n_{6} = \frac{Pr}{2}$; $n_{7} = n_{6} + 1$; $n_{8} = m_{1} + \alpha_{1}$; $n_{9} = m_{2} + \alpha_{1}$

$K_{5} = m_{1}K_{2}$; $K_{6} = m_{1}K_{12}$; $K_{7} = m_{1}K_{13} + m_{3}K_{14}$

$K_{15} = K_{16} - K_{17}$

$l_{9} = 1 + \frac{km_{1}}{Pr} (e^{l} - 1)$; $l_{10} = 1 + \frac{km_{1}}{Pr} (e^{l} - 1)$

$l_{11} = K_{18} e^{-m_{1}} + K_{19} e^{-m_{2}} + K_{20} e^{-m_{3}} + K_{21} e^{-m_{4}}$

$l_{13} = m_{m_{1}}(K_{18} + m_{2}K_{19}) - K_{21}$

$K_{22} = m_{m_{1}} e^{-m_{1}} - 1; l_{14} = 1 + \frac{km_{1}}{Pr} (e^{l} - 1)$

$l_{15} = 1 + \frac{km_{1}}{Pr} (e^{l} - 1)$; $l_{16} = m_{m_{1}} e^{-m_{1}} - 1$

$K_{23} = m_{m_{1}} e^{-m_{1}} + K_{25} e^{-m_{3}} + K_{26} e^{-m_{4}}$

$K_{24} = K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$

$K_{25} = m_{m_{1}} e^{-m_{1}} - 1; l_{23} = K_{26} e^{-m_{3}} + K_{27} e^{-m_{4}}$

$K_{24} = K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$

$P_{1} = -2E \alpha \rho_{m} C_{e} D_{1}$; $P_{1} = -2E \alpha \rho_{m} C_{e} D_{2}$

$K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$

$K_{24} = K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$

$P_{1} = -2E \alpha \rho_{m} C_{e} D_{1}$; $P_{1} = -2E \alpha \rho_{m} C_{e} D_{2}$

$K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$

$K_{24} = K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$

$K_{24} = K_{26} m_{m_{1}} + K_{27} m_{m_{1}} - m_{1} + K_{28}$
\[
\begin{align*}
P_{42} &= -2EcPrS_{12}^2; \quad P_{43} = -2EcPrS_{33}^2, \\
&= 4 - 2Pr + Pr K_{20} ; \quad 1 - Pr + Pr K_{20}.
\end{align*}
\]
\[
P_{44} = K_{s8} + P_{42}; \quad P_{45} = K_{59} + P_{43}.
\]
\[
R_1 = \alpha_1 E_1 + \beta_1 E_2; \quad R_2 = \alpha_2 E_2 - \beta_2 E_1;
\]
\[
R_3 = m_1 m_2 (C_1 K_{18} + C_1 K_{19}); \quad R_4 = z_4 \beta_4 R_2 - z_3 R_1;
\]
\[
R_5 = z_1 \beta_1 R_1 + z_3 R_2; \quad R_6 = \frac{2Ecm C}{z_3^2 \beta_3^2 + z_3};
\]
\[
R_7 = \frac{2Ecm C_2}{z_4^2 \beta_4^2 + z_4^2}; \quad R_8 = \frac{2Ecm C_2}{z_4^2 \beta_4^2 + z_4^2};
\]
\[
R_9 = 2EcP \left( \frac{2Ecm C_2}{z_4^2 \beta_4^2 + z_4^2}; \quad R_{10} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}.
\]
\[
R_{11} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}; \quad R_{12} = \frac{-2Ecm R_3}{K_2 m_3^2 - m_3 + K_{20}}.
\]
\[
R_{13} = C_1 K_{19} + C_3 K_{18}; \quad R_{14} = z_4 \beta_4 E_2 - z_3 E_1;
\]
\[
R_{15} = z_4 \beta_4 E_1 + z_4 E_2; \quad R_{16} = \frac{2Ecm C}{z_4^2 \beta_4^2 + z_4^2};
\]
\[
R_{17} = \frac{-2Ecm C_2}{z_4^2 \beta_4^2 + z_4^2}; \quad R_{18} = z_4 \beta_4 E_1 + z_4 E_2;
\]
\[
R_{19} = \frac{2Ecm C}{z_4^2 \beta_4^2 + z_4^2}; \quad R_{20} = \frac{2Ecm C}{z_4^2 \beta_4^2 + z_4^2};
\]
\[
R_{21} = z_4 \beta_4 E_1 + z_4 E_2; \quad R_{22} = \frac{2EcP}{z_4^2 \beta_4^2 + z_4^2};
\]
\[
R_{23} = \frac{2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}; \quad R_{24} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}.
\]
\[
R_{25} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}; \quad R_{26} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}.
\]
\[
R_{27} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}; \quad R_{28} = R_8 R_4 + R_9 R_14; \quad R_{29} = R_8 E_5 + R_{16} R_{15};
\]
\[
R_{30} = R_8 R_5 + R_9 R_{17}; \quad R_{31} = R_8 R_8 + R_9 R_{18};
\]
\[
R_{32} = K_{27} + R_{10} + R_{23}; \quad R_{33} = K_{28} + R_{11} + R_{24};
\]
\[
R_{34} = K_{29} + R_{12} + R_{25}; \quad R_{35} = K_{25} + R_{26} + R_{36} = K_{26} + R_{27};
\]
\[
R_{37} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}};
\]
\[
R_{38} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}};
\]
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\]
\[
R_{44} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}};
\]
\[
R_{45} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}};
\]
\[
R_{46} = \frac{-2Ecm C_2}{4K_2 m_3^2 - 2m_2 + K_{20}}.
\]
\[
R_{47} = K_{s8} + R_{45}; \quad R_{48} = K_{59} + R_{46}.
\]
\[
S_1 = m_1 C_1 (n_1 D_1 + D_2); \quad S_2 = D_2 m_1 n_1 C_1;
\]
\[
S_3 = m_1 C_2 (n_1 D_1 + D_2); \quad S_4 = D_2 m_1 n_1 C_2;
\]
\[
S_5 = m_1 m_2 (C_2 K_{18} + C_1 K_{19}); \quad S_6 = m_2 m_2 (C_1 K_{18} + C_1 K_{19});
\]
\[
S_7 = m_2 m_2 (C_1 K_{18} + C_1 K_{19}); \quad S_8 = C_4 (m_2 D_3 + D_4); \quad S_9 = C_4 m_2 n_2;
\]
\[
S_{10} = P(n_2 D_3 + D_4); \quad S_{11} = P D_2 m_1; \quad S_{12} = C_4 K_{27}; \quad S_{13} = PK_{27};
\]
\[
v_1 = l_{10} + v_2 = l_{11}; \quad v_3 = l_{12}; \quad v_4 = l_{13}; \quad v_5 = l_{14} - v_6;
\]
\[
v_6 = \frac{V_2}{e_{n_2} + v_4}; \quad v_7 = m_1 + v_5; \quad v_8 = v_4 v_3 + v_6.
\]