THERMAL NON-EQUILIBRIUM MODELING OF NATURAL CONVECTION HEAT TRANSFER ON A VERTICAL PLATE IN A SATURATED POROUS MEDIUM WITH INERTIAL EFFECTS

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ABSTRACT

In the present work, the effect of flow inertia on natural convection heat transfer on a vertical plate in a saturated porous medium using the thermal non-equilibrium model is investigated. The transformed conservation equations of the non-similar boundary layer are solved numerically using an efficient iterative implicit finite-difference method. The numerical results are validated by favorable comparisons with previously published work. Two cases were considered, the uniform heat flux case \((\lambda = 1/3)\) and the natural convection pure Darcy flow adjacent to an isothermal plate \((\lambda = 0)\). Numerical results for the non-similar transformation function, velocity of the fluid and rate of change of velocity function of the fluid as well as the local Nusselt number for the fluid and solid phases are presented graphically and discussed.

KEYWORDS: Natural convection; vertical plate; porous medium; thermal non-equilibrium model.

1. INTRODUCTION

The concept of heat transfer in a porous medium is observed in many important applications such as geophysical systems, heat exchangers, chemical reactors, thermal insulation of buildings, nuclear waste, petroleum resources, drying processes and others. More applications and good understanding of the subject is given in [1-5]. An inspection of the literature reveals that the subject of heat transfer in porous media has gained considerable momentum in the last few decades as many researchers have paid their attention to study various aspects of transport in porous media. Kim and Vafai [6] have studied buoyancy-driven flow about a vertical plate for constant wall temperature and constant heat flux. They found that the heat transfer rate depended on the modified Rayleigh number when the thermal boundary layer thickness is greater than the viscous boundary layer, whereas the heat transfer rate depended on the product of the Rayleigh number and porosity if the thickness of the viscous boundary layer is greater than the thermal boundary layer. Seetharamu and Dutta [7,8] have analyzed free convection in a saturated porous medium adjacent to a non-isothermal vertical impermeable wall and non-uniform heat flux. Cheng and Minkowycz [9] have studied free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dyke. The presence of radiation is generally neglected in the analysis because of the complexity of the problem. Thermal radiation plays a significant role in the overall surface heat transfer where convective heat transfer is small. Chen and Ho [10] have investigated the effects of flow inertia on vertical natural convection in saturated porous medium.

All of the above studies used the thermal equilibrium model between the fluid and the porous medium. However, in general the thermal conductivities of the fluid and the porous medium are different and the heat transfer or absorption from the fluid or the porous medium is different and therefore, the thermal equilibrium model has shortcomings. The objective of this paper is to use a thermal non-equilibrium model to study the natural convection heat transfer on a vertical plate in a saturated porous medium in the presence of porous medium inertial effects. Here, the single energy equation which is used for the thermal equilibrium model is replaced with two energy equations, one for the solid porous medium and another for the fluid. The coupling of these two equations is given by the interfacial heat transfer coefficient. Two energy equation models have been introduced heuristically in the literature [11]. An efficient, iterative, tri-diagonal implicit finite difference method is used to solve the transformed non-similar conservation equations of the boundary layer. The obtained results are presented graphically to illustrate the influences of different physical parameters on the local Nusselt number for the fluid and solid phases, velocity, dimensionless non-similar transformation function and the dimensionless temperature profiles for fluid and solid phases.

2. PROBLEM FORMULATION

Consider the problem of natural convection heat transfer on a vertical plate in a saturated porous medium. In this problem the following assumptions are made:
a) The thermal non-equilibrium model between the fluid and solid porous medium is used.

b) Properties of the fluid and the porous medium are everywhere isotropic and homogeneous.

c) Forchheimer's model of porous medium inertial effects is used in the governing equations of the problem.

d) The Boussinesq approximation is employed.

Taking the above assumptions into consideration, the governing equations are given by

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
\frac{\mu_f}{K} u + \beta \rho_f g \frac{\partial T_f}{\partial x} = -\frac{\partial p'}{\partial x} - \rho_f g \left[ T_f - T_w \right], \quad (2)
\]

\[
\frac{\mu_f}{K} v + \beta \rho_f g \frac{\partial T_f}{\partial y} = -\frac{\partial p'}{\partial y}, \quad (3)
\]

\[
\frac{(\rho c_p_f)}{K} \left( u \frac{\partial^2 T_f}{\partial x^2} + v \frac{\partial^2 T_f}{\partial y^2} \right) = \varepsilon \frac{\partial^2 T_f}{\partial x^2} + \frac{\partial^2 T_f}{\partial y^2} + h \left[ T_s - T_f \right], \quad (4)
\]

\[
[1 - \varepsilon] k_s \frac{\partial^2 T_s}{\partial y^2} + h \left[ T_f - T_s \right] = 0. \quad (5)
\]

Note that, in Equation (7) we have made use of the fact that

\[
u \sqrt{u^2 + v^2} = u^2 \sqrt{1 + (u/v)^2} \approx u^2. \quad (11)
\]

Eliminating \(p'\) from Equations (6) and (7) yields

\[
\frac{\partial T_f}{\partial y} = \frac{\mu_f}{\rho_f \varepsilon K} \frac{\partial u}{\partial y} + \frac{b}{\rho_c} \frac{\partial u^2}{\partial y}. \quad (11)
\]

The appropriate boundary conditions for Equations (6), (9), (10) and (11) are given by

\[
y = 0: \quad v = 0, \quad T_f = T_w = T_m + Ax^k, \quad T_s = T_w = T_m + Ax^k, \quad (12)
\]
\[
y \to \infty: \quad u = 0, \quad T_f = T_m, \quad T_s = T_m,
\]

where \(A\) and \(\lambda\) are constants. In Eq. (12), the fluid and solid phase temperatures are assumed to be power functions of \(x\). By suitable choice of \(\lambda\), the problem can be related to different cases. The case \(\lambda = 0\) represents the natural convection adjacent to an isothermal plate while the case \(\lambda = 1/3\) corresponds to the case uniform heat flux plate (see Chen and Ho [10]).

The following variables are used to transform the \((x,y)\) coordinates to dimensionless \([\xi(x), \eta(x,y)]\) coordinates:

\[
\eta = \frac{y}{x} \frac{Ra_x^{1/2}}{\varepsilon \alpha_f}, \quad \xi = \frac{x}{x} \frac{Ra_x^{1/2}}{\varepsilon \alpha_f} \quad (13)
\]

The coordinate \(\xi(x)\) is so chosen that \(x\) does not appear explicitly in either of the transformed equations or the transformed boundary conditions. In addition, the dimensionless stream function \(\psi(\xi, \eta)\), the dimensionless temperature function for the fluid phase \(\theta_f(\xi, \eta)\) and the dimensionless temperature function for the solid phase \(\theta_s(\xi, \eta)\) are defined, respectively, as

\[
\psi(\xi, \eta) = \frac{\psi(x,y)}{\varepsilon \alpha_f} \frac{Ra_x^{1/2}}{\varepsilon \alpha_f}, \quad \theta_f(\xi, \eta) = \frac{\theta_f(x,y) - T_m}{T_w - T_m}, \quad \theta_s(\xi, \eta) = \frac{\theta_s(x,y) - T_m}{T_w - T_m}, \quad (14)
\]

In Equations (14), \(\alpha_f\) is the thermal diffusivity for the fluid phase, \(\psi(x,y)\) is the stream function which are defined as

\[
u = \frac{\partial \psi}{\partial x}, \quad \alpha_f = \frac{k_f}{(\rho c_p)_f}. \quad (15)
\]

Substituting Equations (13), (14) and (15) into Equations (9), (10) and (11) yields the following dimensionless equations:
\[ \theta'_{F} - F^* = 2\xi F'F^* \]  
\[ \theta'_{F} + \frac{(\lambda + 1)}{2} F \theta'_{F} - \lambda F' \theta_{F} + H[\theta_s - \theta_{F}] = \lambda \xi \left( F \frac{\partial \theta_{F}}{\partial \xi} - \theta_{F} \frac{\partial F}{\partial \xi} \right) \]  
\[ \theta'_{s} + \chi H[\theta_{F} - \theta_{s}] \]  
(16)  
(17)  
(18)

where the primes in these equations denote differentiation with respect to \( \eta \) and

\[ \xi = (Ra_x \times Pr)Fo_{x}, \quad Fo_{x} = Kbc/x, \]

\[ H = (h \times \varepsilon \times Ra_{s}^{-1}), \quad \varepsilon = \frac{ek_{f}}{(1 - e)k_{s}} - Ra_{s}^{-1}, \]

(19)

where \( Pr \) is the Prandtl number, \( Fo_{x} \) is a new dimensionless number (see [10]), \( H \) is the heat transfer coefficient parameter and \( \varepsilon \) is the thermal conductivity ratio parameter.

The transformed boundary conditions become:

\[ \eta = 0: \frac{\lambda + 1}{2} F(\xi, 0) + \lambda \xi \frac{\partial F}{\partial \xi} = 0, \theta_{F}(\xi, 0) = 1, \theta_{s}(\xi, 0) = 1, \]

\[ \eta \to \infty: F(\xi, \infty) = 0, \theta_{F}(\xi, \infty) = 0, \theta_{s}(\xi, \infty) = 0. \]

(20)

Moreover, the velocity components in terms of the new variables, can be expressed by

\[ u = \frac{Ra_{x} \times \varepsilon \alpha_{f} F}{x}, \]

\[ v = \frac{-Ra_{x}^{1/2} \times \varepsilon \alpha_{f} \left[ \frac{\lambda + 1}{2} F + \lambda \xi \frac{\partial F}{\partial \xi} \right] \frac{n}{F^{2}}}{x}. \]

(21)  
(22)

Of special significance for this flow and heat transfer situation are the local wall shear stress \( \tau_w \), the local rate of heat transfer for the fluid phase (\( q_{w,f} \)) and the local rate of heat transfer for the solid phase (\( q_{w,s} \)). These physical quantities can be defined as

\[ \tau_w = \frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{\mu \alpha_{f} \times Ra_{x}^{3/2} F(\xi, 0)}{x^2} \]  

\[ (q_{w,f})_{f} = -k_{f} \frac{\partial T_f}{\partial y} \bigg|_{y=0} = -k_{f} A^{3/2} \left( \frac{\rho K_{B} \beta}{\mu \alpha_{f}} \right)^{1/2} x^{(3\lambda-1)/2} \theta_{f}(\xi, 0) \]  

\[ (q_{w,s})_{s} = -k_{s} \frac{\partial T_s}{\partial y} \bigg|_{y=0} = -k_{s} A^{3/2} \left( \frac{\rho K_{B} \beta}{\mu \alpha_{s}} \right)^{1/2} x^{(3\lambda-1)/2} \theta_{s}(\xi, 0). \]

(23)  
(24)  
(25)

The local skin-friction coefficient and the local Nusselt number for the fluid and solid phases may be written as

\[ C_{f} = \frac{\tau_w}{\mu \alpha_{f} / x^2} = Ra_{x}^{3/2} F(\xi, 0) \]  

\[ Nu_{f} / Ra_{x}^{1/2} = -\theta'_{f}(\xi, 0) \]  

\[ Nu_{s} / Ra_{x}^{1/2} = -\theta'_{s}(\xi, 0) \]

(26)  
(27)  
(28)

3. Numerical Method and Validation

The heat transfer problem represented by Equations (16) through (18) is nonlinear and, therefore, must be solved numerically. The standard implicit finite-difference method discussed by Blottner [12] has proven to be adequate and gives accurate results for such equations. For this reason, it is employed in the present work and graphical results based on this method will be presented subsequently.

Equations (16)-(18) are discretized using three-point central difference formulae. The \( \eta \) direction is divided into 196 nodal points and a variable step size is used to account for the sharp changes in the variables in the region close to the surface where viscous effects dominate. The initial step size used is \( \Delta \eta = 0.001 \) and the growth factor \( K = 1.037 \) such that \( \Delta \eta_n = K \Delta \eta_{n-1} \) (where the subscript \( n \) is the number of nodes minus one). This gives \( \eta_{\text{max}} \approx 35 \) which represents the edge of the boundary layer at infinity. A two-point backward-difference formula is used for the first derivatives with respect to \( \xi \). A constant step size of 0.01 is used in the \( \xi \) direction. The problem is solved as an initial value problem with \( \xi \) playing the role of time. At each line of constant \( \xi \), the ordinary differential equations are converted into linear algebraic equations that are solved by the Thomas algorithm discussed by Blottner [12]. Iteration is employed to deal with the nonlinear nature of the governing equations. The convergence criterion employed in this work was based on the relative difference between the current and the previous iterations. When this difference or error reached 10\(^{-5}\), then the solution was assumed converged and the iteration process was terminated. This process is continued until the final desired \( \xi \) is reached. In order to check the accuracy of the numerical method, the local Nusselt number of the fluid phase \( Nu_{f} / Ra_{x}^{1/2} \) is compared with those reported earlier by Chen and Ho [10] using the thermal-equilibrium model. As shown in Table 1, the present results are found to be in excellent agreement with the results of Chen and Ho [10].

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4. RESULTS AND DISCUSSION

Numerical computations are carried out and a parametric study is performed to illustrate the influence of the physical parameters on the non-similar transformation function, velocity, rate of change in the velocity of the fluid and the fluid- and solid-phase temperatures as well as the local Nusselt numbers for the fluid and solid phases. The results of this parametric study are shown in Figs. 1-12.

Figures 1-4 show the inertia effects on the non-similar transformation function \( F(\xi, \eta) \), velocity of the
fluid $F(\xi, \eta)$, rate of change in the velocity of the fluid represented by $F'(\xi, \eta)$, temperature of the fluid phase $\theta_f(\xi, \eta)$ and the temperature of the solid phase $\theta_s(\xi, \eta)$ for the case of uniform heat flux ($\lambda = 1/3$), respectively. It is clear that as the flow inertia parameter $\xi$ increases, this causes a decrease not only in the fluid velocity but also in the temperatures of the fluid and solid phases.

Fig. 1. Inertia effects on the non-similar transformation $F(\xi, \eta)$ for the case of uniform heat flux, $\lambda = 1/3$.

Fig. 2. Inertia effects on the velocity of the fluid for the case of uniform heat flux, $\lambda = 1/3$.

Fig. 3. Inertia effects on the rate of change of velocity for the case of uniform heat flux, $\lambda = 1/3$.

Fig. 4. Inertia effects on the temperature of the fluid and solid phases

In addition, the effects of heat transfer coefficient parameter $H$ on the local Nusselt number of the fluid phase $\text{Nu}_f \text{Ra}^{1/2}_x$ and solid phase $\text{Nu}_s \text{Ra}^{1/2}_x$ for the case of uniform heat flux ($\lambda = 1/3$) are plotted in Fig. 5. From this figure, we can find that the rate of heat transfer for the fluid phase is higher than the rate of heat transfer for the solid phase. Also, increasing the value of the heat transfer coefficient parameter $H$ results in increases in both of the local Nusselt numbers of the fluid phase ($\text{Nu}_f \text{Ra}^{1/2}_x$) and the solid phase ($\text{Nu}_s \text{Ra}^{1/2}_x$). Further, observing the effects of the length of the plate represented by the inertia parameter $\xi(x)$, we can find that the lower part of the plate is hotter than the upper part. On the other hand, the effects of thermal conductivity ratio $\chi$ on the local Nusselt number of the solid phase $\text{Nu}_s \text{Ra}^{1/2}_x$ for the case of uniform heat flux ($\lambda = 1/3$) are depicted in Fig. 6. It must be noted that we can increase the rate of heat transfer for the solid phase by increasing the thermal conductivity ratio $\chi$. All these behaviors are clearly presented in Figs. 1-6.
Fig. 6. Effects of thermal conductivity ratio $\chi$ on the local Nusselt number of the solid phase $\text{Nu}_s \text{Ra}_x^{-1/2}$ for the case of uniform heat flux, $\lambda = 1/3$.

With the help of Figs. 7-10, we can describe the inertia effects on the non-similar transformation function $F(\xi, \eta)$, velocity of the fluid $F'(\xi, \eta)$, the rate of change in the velocity of the fluid represented by $F''(\xi, \eta)$, temperature of the fluid phase $\theta_f(\xi, \eta)$ and the temperature of the solid phase $\theta_s(\xi, \eta)$ for the case of pure Darcy flow ($\lambda = 0$), respectively. From these figures we observe that the values of non-similar transformation function $F(\xi, \eta)$, velocity of the fluid $F'(\xi, \eta)$, the rate of change in the velocity of the fluid represented by $F''(\xi, \eta)$, temperature of the fluid phase $\theta_f(\xi, \eta)$ and the temperature of the solid phase $\theta_s(\xi, \eta)$ are smaller than those corresponding to the case of uniform heat flux represented by $\lambda = 1/3$. This means that decreasing the value of the surface condition parameter $\lambda$ leads to a decay in the fluid motion.

Fig. 7. Inertia effects on the non-similar transformation $F(\xi, \eta)$ for the case of pure Darcy flow, $\lambda = 0$.

Fig. 8. Inertia effects on the velocity of the fluid for the case of pure Darcy flow, $\lambda = 0$.

Fig. 9. Inertia effects on the rate of change of velocity for the case of pure Darcy flow, $\lambda = 0$.

Fig. 10. Inertia effects on the temperature of the fluid and solid phases for the case of pure Darcy flow, $\lambda = 0$.

In addition, Fig. 11 displays the effects of the heat transfer coefficient parameter $H$ on the local Nusselt number of the fluid phase $\text{Nu}_f \text{Ra}_x^{-1/2}$ and solid phase $\text{Nu}_s \text{Ra}_x^{-1/2}$ for the case of pure Darcy flow, $\lambda = 0$. It is clear from this figure that, an increase in the heat transfer coefficient parameter $H$ results in increasing in the local Nusselt number of the
fluid phase $\text{Nu}_f \text{Ra}_x^{-1/2}$ and the solid phase $\text{Nu}_s \text{Ra}_x^{-1/2}$. Furthermore, the effects of the thermal conductivity ratio $\chi$ on the local Nusselt number of the solid phase $\text{Nu}_s \text{Ra}_x^{-1/2}$ for the case of pure Darcy flow ($\lambda = 0$) are plotted in Fig. 12. It is observed that, as the thermal conductivity ratio $\chi$ increases, the local Nusselt number for the solid phase increases whereas, it decreases as the inertia parameter $\xi$ increases. All of the above trends are clearly depicted in Figs. 7-12.

**Fig. 11.** Effects of heat transfer coefficient parameter $H$ on the local Nusselt number of the fluid phase $\text{Nu}_f \text{Ra}_x^{-1/2}$ and solid phase $\text{Nu}_s \text{Ra}_x^{-1/2}$ for the case of pure Darcy flow, $\lambda = 0$.

**Fig. 12.** Effects of thermal conductivity ratio $\chi$ on the local Nusselt number of the solid phase $\text{Nu}_s \text{Ra}_x^{-1/2}$ for the case of pure Darcy flow, $\lambda = 0$.

5. CONCLUSIONS

Thermal non-equilibrium modeling of natural convection heat transfer on a vertical plate in saturated porous medium with the effects of flow inertia is studied. The governing equations are obtained and transformed into a non-similar form. The non-similar equations are solved numerically by an efficient tri-diagonal implicit finite-difference method. Comparisons of the present results with previously published results are found to be in excellent agreement. From the results of the problem, we observed that

I. Increasing the value of the surface condition parameter $\lambda$ leads to increases in the local Nusselt number for fluid phase $\text{Nu}_f / \text{Ra}_x^{1/2}$ and the local Nusselt number for solid phase $\text{Nu}_s / \text{Ra}_x^{1/2}$.

II. The porous medium inertia effects produced reductions in the fluid velocity and the heat transfer characteristics.

III. The rate of heat transfer for the solid phase is reduced by increasing the thermal conductivity ratio.

REFERENCES