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Volume 8, Number 3, October 1995
UNSTEADY HYDROMAGNETIC TWO-PHASE PIPE FLOW

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Continuum equations governing unsteady flow of a particulate suspension in an electrically conducting fluid through a circular pipe in the presence of a uniform transverse magnetic field are formulated. Two different applied pressure gradients (constant and oscillating) cases are considered. The governing equations of motion are nondimensionalized and solved in closed form in terms of Bessel functions. Numerical evaluations of the exact solutions are performed and graphical results for the fluid-phase volumetric flow rate, the particle-phase volumetric flow rate, and the fluid-phase skin friction coefficients are presented and discussed to show the effects of the particle loading, the inverse Stokes number, and the Hartmann magnetic number on the solutions.


INTRODUCTION

The flow of an electrically conducting fluid in circular pipes under transverse magnetic fields occurs in magnetohydrodynamic (MHD) generators, pumps, accelerators, and flowmeters. The possible presence of solid particles such as ash or soot in combustion MHD generators and plasma MHD accelerators and its effect on the performance of such devices led to studies of particulate suspensions in conducting fluids in the presence of magnetic fields. The single-phase flow of conducting fluids in pipes with circular cross sections has been investigated by many authors (see, for instance, Shercliff, 1956, and Gold, 1962), Dube and Sharma (1975) and Ritter and Peddieson (1977) reported solutions for unsteady dusty-gas flow in a circular pipe in the absence of a magnetic field. It is of interest in this paper to generalize the axisymmetric circular pipe flow solutions reported by Dube and Sharma (1975) and Ritter and Peddieson (1977) for a particulate suspension in an electrically conducting fluid in the presence of a transverse magnetic field. Both the fluid and particle phases are assumed to be incompressible and the particle volume fraction is assumed to be constant and finite. In addition, the particle phase is assumed to be pressureless and electrically nonconducting. The magnetic Reynolds's number is assumed to be small and the induced magnetic field is neglected. The purpose of this paper is to obtain closed-form transient solutions for hydromagnetic two-phase particulate suspension flow in circular pipes. This will be done for both constant and oscillating pressure gradients applied along the flow direction.

Figure 1.
Problem Definition

GOVERNING EQUATIONS

Consider an unsteady, laminar, hydromagnetic, fully developed, axisymmetric flow of a particle/liquid suspension in a horizontal circular pipe due to the action of an arbitrary time varying pressure gradient. A uniform transverse magnetic field is applied normal to the flow direction (see Fig. 1). The Hall effect of magnetohydrodynamics is neglected in this problem. The governing equations for this study are based on the conservation laws of mass and linear momentum of both phases. In this work, it is assumed that both phases are treated as two interacting continua (see, for instance, Marblo, 1970, and Ishii, 1975). The interaction between the phases is restricted to the drag force which is modeled by Stoke's linear drag theory. Under these assumptions, the governing equations of motion can be written as

\[
\rho \frac{\partial \mathbf{V}}{\partial t} = -\nabla \mathbf{\tau} + \frac{1}{\Pr} \nabla \mathbf{\tau} + \mathbf{V} \left( \nabla \mathbf{\tau} - \nabla \mathbf{V} \right) - \nabla \mathbf{B} \cdot \mathbf{V}
\]

where \( \rho \) is the density, \( \mathbf{V} \) is the velocity, \( \mathbf{\tau} \) is the stress tensor, and \( \mathbf{B} \) is the magnetic field. (1)
\[ \rho_p \partial_t V_p = \rho_p \left( V - V_p \right) / \tau_v \]  
\[ \rho_p \partial_t V_p = \rho_p \left( V - V_p \right) / \tau_v \]  
\[ F(\eta, 0) = 0, \quad F_p(\eta, 0) = 0, \]  
\[ F(0, \tau) = \infty, \quad F(1, \tau) = 0 \]  

where \( t \) is time, \( r \) is the distance in the radial direction, \( \phi \) is the particle volume fraction, \( V \) is the fluid-phase velocity, \( V_p \) is the particle-phase velocity, \( \rho_p \) is the fluid density, \( \rho_p \) is the fluid density, \( \partial_r F_p \) is the fluid pressure gradient, \( \mu \) is the fluid dynamic viscosity, \( \tau_v \) is the momentum relaxation time (time necessary to reduce the slip velocity between the phases by \( e^1 \) of its initial value), \( \sigma \) is the fluid electrical conductivity, and \( \beta_0 \) is the magnetic induction.

Eqs. 1 and 2 are solved subject to the following initial and boundary conditions:

\[ V(r, 0) = 0, \quad V_p(r, 0) = 0, \]  
\[ V(0, \tau) = \infty, \quad V(a, \tau) = 0 \]  

where \( a \) is the pipe radius.

Eqs. 1 through 3 constitute an initial-value problem. Non-dimensionalization of the governing equations and conditions can be accomplished by using the following parameters:

\[ r = a \eta, \quad t = \eta \tau, \quad \rho_p = \frac{\kappa \rho_p \eta}{\eta} / \phi, \]  
\[ \rho_p = \frac{\kappa \rho_p \eta}{\eta} / \phi, \]  
\[ \partial_r F_p = G_0 G(\tau), \]  
\[ V(r, \tau) = G_0 G(\tau) F(\eta, \tau), \]  
\[ V_p(r, \tau) = G_0 G(\tau) F_p(\eta, \tau) \]  

where \( \eta \) is the dimensionless radial distance, \( r \) is the dimensionless time, \( \kappa \) is the particle loading, \( G_0 \) is a constant, \( G \) is the applied pressure gradient, and \( F \) and \( F_p \) are the dimensionless fluid- and particle-phase velocities, respectively. After performing the mathematical operations, the resulting dimensionless equations and conditions can be written as

\[ \partial_r F = \frac{\partial_r F_p}{\eta} + \frac{1}{\kappa \rho_p \eta} F + \kappa \rho_p \left( F_p - F \right) - M^2 F + G(\tau) \]  
\[ \partial_r F_p = \alpha \left( F - F_p \right) \]  

(5)  
(6)  

where \( \alpha = \frac{\kappa \rho_p}{\eta} \) and \( M = \sqrt{\kappa \rho_p / \eta} \) are the inverse Stoke's number and the Hamann number, respectively.)

RESULTS

Eqs. 5 through 7 represent an initial-value problem which can be solved in closed form by the separation of variables method. This is possible because of the idealization of assumptions made in this problem. It can be assumed that the solutions for \( F \) and \( F_p \) take on the form

\[ F(\eta, \tau) = \sum_{n=1}^{\infty} H_n(\tau) J_0(\lambda_n \eta), \]  
\[ F_p(\eta, \tau) = \sum_{n=1}^{\infty} H_p(\tau) J_0(\lambda_n \eta) \]  

(9)  

where \( \lambda_n \) are the roots of the equation \( J_0(\lambda_n \eta) = 0 \) (\( \lambda_0 \) being the zero-order Bessel function of the first kind) and \( H_n \) and \( H_p \) are functions to be determined. Substituting Eqs. 5 and their derivatives into Eqs. 5 and 6 (with the non-homogeneous part of these equations represented by a series in \( J_0(\lambda_n \eta) \), multiplying by \( J_0(\lambda_m \eta) \) (to take advantage of the orthogonality property of Bessel functions), and then integrating with respect to \( \eta \) from 0 to 1 yield

\[ H_n + \left( \lambda_n^2 + \kappa \rho_p \eta \right) H_n - \kappa \rho_p \eta H_{pn} = b_n G(\tau) \]  
\[ H_{pn} + \alpha \left( H_{pn} - H_n \right) = 0 \]  

where \( \lambda_n \) are the roots of the equation \( J_0(\lambda_n \eta) = 0 \) (\( \lambda_0 \) being the zero-order Bessel function of the first kind) and \( H_n \) and \( H_p \) are functions to be determined. Substituting Eqs. 5 and their derivatives into Eqs. 5 and 6 (with the non-homogeneous part of these equations represented by a series in \( J_0(\lambda_n \eta) \), multiplying by \( J_0(\lambda_m \eta) \) (to take advantage of the orthogonality property of Bessel functions), and then integrating with respect to \( \eta \) from 0 to 1 yield

\[ H_n + \left( \lambda_n^2 + \kappa \rho_p \eta \right) H_n - \kappa \rho_p \eta H_{pn} = b_n G(\tau) \]  
\[ H_{pn} + \alpha \left( H_{pn} - H_n \right) = 0 \]  

Fluid / Particle Separation Journal 205
Vol. 8, No. 3, October 1995
where a dot denotes ordinary differentiation with respect to $t$ and $b_n = 2/(\lambda_0\alpha_1(\lambda_n))$, ($\alpha_1$ being the first-order Bessel function of the first kind). Combining Eqs. 10 and 11 gives

$$H_n + \left(\lambda_n^2 + \alpha(1 + \kappa) + M^2\right)H_n + \alpha\lambda_n^2 + M^2 = b_n \left(G(t) + \alpha G(t)\right)$$

(12)

Constant Pressure Gradient Solutions

For a constant applied pressure gradient $G(t)=1$. Substituting this into Eq. 12 gives a linear, ordinary, non-homogeneous, differential equation whose solution subject to $H_0(0) = 0$ and $H_{np}(0) = 0$ can be obtained by the usual method of solving such equations and is shown to be

$$H_n = c_1\exp(s_1t) + c_2\exp(s_2t) + b_n\left(\frac{\lambda_n^2 + M^2}{\alpha}\right)$$

(13)

where $s_1$ and $s_2$ are the roots of the equation

$$s^2 + \left(\lambda_n^2 + \alpha(1 + \kappa) + M^2\right)s + \alpha\lambda_n^2 + M^2 = 0$$

(14)

The solution for $H_{np}$ can be obtained from Eq. 11 to give

$$c_1 = b_n\left(s_2 + \lambda_n^2 + M^2\right) / \left(\left(\lambda_n^2 + M^2\right)\left(s_2 - s_1\right)\right)$$

$$c_2 = b_n\left(s_1 - \lambda_n^2 - M^2\right) / \left(\left(\lambda_n^2 + M^2\right)\left(s_2 - s_1\right)\right)$$

(15)

$$H_{np} = \frac{1}{\left(\alpha\alpha_1\right)} c_1 \left(s_1 + \lambda_n^2 + \kappa\alpha + M^2\right) \exp(s_1t) + c_2 \left(s_2 + \lambda_n^2 + \kappa\alpha + M^2\right) \exp(s_2t) + b_n / \left(\lambda_n^2 + M^2\right)$$

(16)

The corresponding expression for $Q$, $Q_p$, and $C$ can be written as

$$Q = 4\pi \sum_{n=1}^{\infty} H_n / \left(b_n \lambda_n^2\right)$$

$$Q_p = 4\pi \sum_{n=1}^{\infty} H_{np} / \left(b_n \lambda_n^2\right)$$

$$C = \sum_{n=1}^{\infty} H_n / b_n$$

(17)

It should be noticed that Eqs. 17 take into account the symmetry of the problem.

The transient behavior of the fluid-phase volumetric flow rate $Q$, the particle-phase volumetric flow rate $Q_p$, and the fluid-phase skin friction coefficient $C$ is presented graphically in Figs. 2 through 10. These figures are obtained by numerically evaluating Eqs. 17 for various values of $\kappa$, $\alpha$, and $M$, and are chosen to illustrate the influence of these parameters on the solutions.
Figure 10. The Effect of M on Fluid-Phase Skin-Friction Coefficient Time History

Figs. 2 through 4 present the time histories of $Q$, $Q_p$, and $C$ for different values of the particle loading $k$, respectively. Increasing the particle loading in the pipe has the tendency to slow down the motion of the suspension, which results in decreasing $Q$, $Q_p$, and $C$ as depicted in Figs. 2 through 4, respectively. It can also be seen from Figs. 2 through 4 that $Q$, $Q_p$, and $C$ increase as time progresses until steady-state conditions are achieved at sufficiently large values of time. The influence of the inverse Stoke's number $\alpha$ on the same physical parameters is illustrated in Figs. 5 through 7. As $\alpha$ increases, the interphase momentum transfer due to the drag mechanism increases, causing the fluid-phase velocity to decrease and the particle-phase velocity to increase. This results in decreases in the values of $Q$ and $C$, and increases in the values of $Q_p$ as shown in these figures. It should be mentioned that an examination of Figs. 5 and 7 shows that $Q$ and $C$ do not have an orderly progression since part of the curves associated with $\alpha = 1$ lie below the curve for $\alpha = 100$. Indeed, the value $\alpha = 0.1$ appears to be a critical value as it rises abruptly and then approaches the steady-state value in a longer period of time. A similar phenomenon has been observed by Gold (1962) in his study regarding the effect of Hartmann number on velocity distribution in magneto-hydrodynamic pipe flow.

Figs. 8 through 10 depict the influence of the Hartmann number $M$ on $Q$, $Q_p$, and $C$, respectively. Increases in the values of $M$ cause the velocities of both the fluid and particle phases to decrease. This results in a decrease in the flow rates of both phases. This is clearly shown in Figs. 8 and 9. Also, the slope of the fluid-phase velocity profile at the pipe surface decreases by increasing $M$ causing the skin-friction coefficient $C$ to decrease as observed in Fig. 10. It should be mentioned that the results for $M=0$ are consistent with those reported by Dube and Sharma (1975) and Ritter and Peddeson (1977).

Furthermore, when $k$ is equated to zero, the classical results of Batchelor (1967) for fluid flow in pipes is recovered.

**Oscillating Pressure Gradient Solutions**

Assuming that oscillating pressure gradient, $G(t)$, is general sinusoidal function of the form

$$G(t) = G_2 \sin(\omega t) + G_3 \cos(\omega t)$$

where $\omega$ (a constant) is the circular frequency of oscillation, and $G_2$ and $G_3$ are constants and substituting into Eq. 12 results in

$$H_n + \left( \lambda_n^2 + \alpha (1+\kappa) + M^2 \right) H_n + \alpha \left( \lambda_n^2 + M^2 \right) H_n = b_n \left( (G_x \alpha - G_z \omega) \sin(\omega t) + (G_z \alpha + G_y \omega) \cos(\omega t) \right)$$

The general solution to the above equation can be shown to be

$$H(n) = c_3 \exp(s_3 t) + c_4 \exp(s_4 t) + c_5 \sin(\omega t) + c_6 \cos(\omega t)$$

where

$$c_3 = (X_1 Z_1 + Y_1 Z_2) / (X_1^2 + Y_1^2),$$
$$c_4 = (X_1 Z_2 + Y_1 Z_1) / (X_1^2 + Y_1^2),$$
$$X_1 = \alpha \left( M^2 + \lambda_n^2 \right) - \omega^2,$$
$$Y_1 = \left( M^2 + \lambda_n^2 + \alpha (1+\kappa) \right) \omega,$$
$$Z_1 = b_n (G_x \alpha - G_z \omega),$$
$$Z_2 = b_n (G_z \alpha + G_y \omega).$$

Application of the initial conditions gives

$$c_4 = (c_6 s_2 - c_5 s_1 - b_y G_y) / (s_2 - s_1),$$
$$c_3 = -c_4 - c_6.$$

For this case the appropriate solution for $H_{pm}$ takes the form

$$H_{pm} = 1(\alpha \lambda)$$

$$= \left( \begin{array}{c}
\left( \begin{array}{c}
c_1 (s_1 + \lambda_n^2 + \kappa \alpha + M^2) \exp(s_1 t) + \\
c_2 (s_2 + \lambda_n^2 + \kappa \alpha + M^2) \exp(s_2 t) + \\
(c_4 \omega + c_6 (M^2 + \lambda_n^2 + \kappa \alpha) - b_y G_y) \cos(\omega t) + \\
(c_5 (M^2 + \lambda_n^2 + \kappa \alpha) - c_6 \omega - b_y G_y) \sin(\omega t)
\end{array} \right)
\end{array} \right)$$

Fluid / Particle Separation Journal Vol. 8, No. 3, October 1995
Figure 11. The Effect of $\kappa$ on Fluid-Phase Volume Flow Rate Time History

Figure 12. The Effect of $\kappa$ on Particle-Phase Volume Flow Rate Time History

Figure 13. The Effect of $\kappa$ on Fluid-Phase Skin-Friction Coefficient Time History

Figure 14. The Effect of $M$ on Fluid-Phase Volume Flow Rate Time History

Figure 15. The Effect of $M$ on Particle-Phase Volume Flow Rate Time History

Figure 16. The Effect of $M$ on Fluid-Phase Skin-Friction Coefficient Time History
The calculations of the volumetric flow rates for both phases and the skin-friction coefficient can be performed by substituting the solutions for \( H_2 \) and \( H_{pm} \) into the expressions given in Eqs. 17.

Some results for \( Q, Q_p \), and \( C \) based on the closed-form solutions for the case of oscillating pressure gradient are presented in Figs. 11 through 16. In Figs. 11 through 13, the influence of \( k \) on these parameters is shown. These and subsequent graphs are plotted about 3 times in time units to show the uniform oscillatory variations which occur after the exponential terms in \( E_2 \) and \( E_{pm} \) decay. As mentioned before, increasing \( k \) causes decreases in the values of \( Q, Q_p \), and \( C \). In fact, for \( k=100 \) the volumetric flow rates are small and the pipe is almost clogged by the presence of particles. These observations are evident from Figs. 11 through 13.

Finally, Figs. 14 through 16 present the variations in \( Q, Q_p \), and \( C \) as the strength of the magnetic field is altered, respectively. The applications of a transverse magnetic field causes a force that acts opposite to the motion of the suspension which results in slowing of its motion in the pipe. As the strength of the applied magnetic field is increased, the influence of this force becomes more pronounced and a greater decrease in the velocities of both phases. This yields a decrease in the values of \( Q, Q_p \), and \( C \) as is clearly shown in Figs. 14 through 16. In comparison with the results presented for constant pressure gradient, it can be concluded that the maximum values of \( Q, Q_p \), and \( C \) are less than the steady-state values for constant pressure gradient. This is expected since \( G' \) is always less than or equal to unity (because of the choice of \( G_2, G_{pm} \), and \( C \)) for the present results. In the absence of magnetic field, the present solutions are in excellent agreement with those given by Ritter (1976).

CONCLUSION

The transient flow of a particulate suspension in an electrically conducting fluid in a circular pipe with an applied transverse magnetic field is studied for both cases of constant and oscillating pressure gradients. The governing equations of motion are derived, nondimensionalized, and solved in closed form. Numerical evaluations of the exact solutions are performed and graphical results for the fluid-phase volumetric flow rate, the particle-phase volumetric flow rate, and the fluid-phase skin friction coefficient are presented and discussed to show the effects of the particle loading, the inverse Stokes number, and the Hartmann magnetic number on the solutions. While comparisons with previously published theoretical work on this problem are performed, so experimental data were found to check the validity of the assumptions made. It is hoped that the results reported herein will serve as a check for further theoretical modeling and a stimulus for experimental work on this problem.

ACKNOWLEDGMENTS

The author wishes to acknowledge the Research Administration at Kuwait University for funding this work under the contract EPM082. Thanks are also due to Engineer Khalil Khanafar for preparing the graphs for this manuscript.

LITERATURE