CONVECTIVE TRANSPORT IN A NANOFLUID SATURATED POROUS LAYER WITH CROSS DIFFUSION AND VARIATION OF VISCOSITY AND CONDUCTIVITY

J. C. Umavathi,1 Ali J. Chamkha,2,* & Monica B. Mohite1

1Department of Mathematics, Gulbarga University, Gulbarga 585 106 Karnataka, India
2Mechanical Engineering Department, Prince Mohammad Bin Fahd University (PMU) P.O. Box 1664, Al-Khobar 31952, Kingdom of Saudi Arabia

*Address all correspondence to Ali J. Chamkha E-mail: a chamkha@pmu.edu.sa

Original Manuscript Submitted: 7/1/2014; Final Draft Received: 1/5/2015

The effect of thermal conductivity and viscosity on linear and nonlinear stability in a horizontal porous medium saturated by a nanofluid has been investigated. The Darcy model has been used for the porous medium, while nanofluid incorporates the effects of Brownian motion along with thermophoresis. In conjunction with the Brownian motion, the nanoparticle fraction becomes stratified, and hence the viscosity and the conductivity are stratified. The linear stability analysis is based on the normal mode technique, while for nonlinear analysis minimal representation of the truncated Fourier series analysis involving only two terms has been used. It is found that for stationary convection Lewis number, the modified diffusivity ratio, viscosity ratio, and conductivity ratio have a stabilizing effect while nanoparticle concentration Rayleigh number and porosity destabilize the system. For oscillatory convection we observe that the thermal capacity ratio, viscosity ratio, and conductivity ratio stabilize the system whereas nanoparticle concentration Rayleigh number, Lewis number, and porosity destabilize the system. For steady finite amplitude motions, the heat and mass transport increases with increase in the values of nanoparticle concentration Rayleigh number, while the heat and mass transport decreases with increase in the values of nanoparticle concentration Rayleigh number, Lewis number, viscosity ratio, and conductivity ratio. The mass transport increases with increase in modified diffusivity ratio. We also study the effect of time on transient Nusselt number and Sherwood number which are found to be oscillatory when time is small. However, when time becomes very large both the transient Nusselt and Sherwood values approach their steady state values.

KEY WORDS: nanofluid, conductivity variation, viscosity variation, Brownian motion, thermophoresis

1. INTRODUCTION

One of the most significant scientific challenges in the industrial area is cooling, which applies to many diverse productions, including microelectronics, transportation, and manufacturing. Technological developments such as microelectronics devices are driving increased thermal loads, requiring advances in cooling.

Conventional methods leading to increased heat transfer rates, such as extended surfaces and microchannels, have the disadvantage of increasing the required pumping power of the cooling fluid, or the use of microfluids poses problems in terms of gravity settling, clogging, etc. The innovative concept of “nanofluids”—heat transfer fluids consisting of suspended of nanoparticles—has been proposed for these challenges.
### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_B$</td>
<td>Brownian diffusion coefficient (m$^2$/s)</td>
</tr>
<tr>
<td>$D_T$</td>
<td>thermophoretic diffusion coefficient (m$^2$/s)</td>
</tr>
<tr>
<td>$H$</td>
<td>dimensional layer depth (m)</td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity of the nanofluid (W/m·K)</td>
</tr>
<tr>
<td>$k_m$</td>
<td>overall thermal conductivity of the porous medium saturated by the nanofluid (W/m·K)</td>
</tr>
<tr>
<td>$K$</td>
<td>permeability (m$^2$)</td>
</tr>
<tr>
<td>$Le$</td>
<td>Lewis number</td>
</tr>
<tr>
<td>$N_A$</td>
<td>modified diffusivity ratio</td>
</tr>
<tr>
<td>$N_B$</td>
<td>modified particle-density increment</td>
</tr>
<tr>
<td>$p^*$</td>
<td>pressure (Pa)</td>
</tr>
<tr>
<td>$Ra_T$</td>
<td>thermal Rayleigh-Darcy number</td>
</tr>
<tr>
<td>$Ra_M$</td>
<td>basic-density Rayleigh number</td>
</tr>
<tr>
<td>$Ra_N$</td>
<td>concentration Rayleigh number</td>
</tr>
<tr>
<td>$t^*$</td>
<td>time (s)</td>
</tr>
<tr>
<td>$T^*$</td>
<td>nanofluid temperature (K)</td>
</tr>
<tr>
<td>$T$</td>
<td>dimensionless temperature, $(T^* - T_{w}^<em>)/(T_b^</em> - T_{w}^*)$</td>
</tr>
<tr>
<td>$T_{w}^*$</td>
<td>temperature at the upper wall (K)</td>
</tr>
<tr>
<td>$T_b^*$</td>
<td>temperature at the lower wall (K)</td>
</tr>
<tr>
<td>$(u^<em>, v^</em>, w^*)$</td>
<td>dimensionless Darcy velocity components</td>
</tr>
<tr>
<td>$v$</td>
<td>nanofluid velocity (m/s)</td>
</tr>
<tr>
<td>$(x, y, z)$</td>
<td>dimensionless Cartesian coordinate</td>
</tr>
</tbody>
</table>

### Greek Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_f$</td>
<td>thermal diffusivity of the fluid (m$^2$/s)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>thermal volumetric coefficient (K$^{-1}$)</td>
</tr>
<tr>
<td>$m$</td>
<td>viscosity variation parameter</td>
</tr>
<tr>
<td>$k$</td>
<td>conductivity variation parameter</td>
</tr>
<tr>
<td>$\mu$</td>
<td>viscosity of the fluid</td>
</tr>
<tr>
<td>$\rho$</td>
<td>fluid density</td>
</tr>
<tr>
<td>$\rho_p$</td>
<td>nanoparticle mass density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>thermal capacity ratio</td>
</tr>
<tr>
<td>$\phi^*$</td>
<td>nanoparticle volume fraction</td>
</tr>
<tr>
<td>$\phi$</td>
<td>relative nanoparticle volume fraction, $(\phi^* - \phi_0^<em>)/(\phi_1^</em> - \phi_0^*)$</td>
</tr>
</tbody>
</table>

### Superscripts

- $*$ dimensional variable
- $'$ perturbed variable

### Subscripts

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>basic solution</td>
</tr>
<tr>
<td>$f$</td>
<td>fluid</td>
</tr>
<tr>
<td>$p$</td>
<td>particle</td>
</tr>
</tbody>
</table>

Maxwell (1873) was the first presenter of a theoretical basis to predict a suspension’s effective conductivity about 140 years ago and his theory was applied from millimeter- to micrometer-sized particle suspensions, but Choi and Eastman (1995) introduced the novel concept of nanofluids by applying the unique properties of nanofluids at the annual meeting of the American Society of Mechanical Engineers in 1995.

A nanofluid is a fluid produced by dispersion of metallic or nonmetallic nanoparticles or nanofibers with a typical size of less than 100 nm in a liquid. These nanofluids can be employed to cool the pipes exposed to such high temperature of the order 100–350°C, while extracting the geothermal energy. Further, when drilling they can also be used as coolants for the machinery and equipment working in high-friction and high-temperature environments.

In the petroleum industry also, nanofluids can be used as coolants or as drilling fluids. Also in the above fields, we come across porous media in the form of rocks inside the Earth’s crust, which is being affected by the rotational component of the Earth’s spin on its axis.

Many researchers have studied the various characteristics of fluid flow and heat transfer behavior of nanofluids over the past one and a half decades, like Masuda et al. (1993), Eastman et al. (2001), and Das et al. (2003), to name a few, and found that enhanced heat transfer coefficients were obtained using nanofluids.

Convection in the porous medium has long been a topic of interest due to its application in fields including food and chemical processes, rotating machineries such as nuclear reactors, the petroleum industry, biomechanics, and geophysical problems, and thus has been studied by...
many researchers, including Horton and Rogers (1945), Rudraiah and Malashetty (1986), Malashetty (1993), and Vafai (2005). Sharma and Singh (2014) have studied the double-diffusive convective instability in a thin layer of a magnetic nanofluid, heated and salted from below and saturating a porous medium within the framework of linear stability theory. The double-diffusive convection in a horizontal layer of nanofluid under rotation in a porous medium has been presented by Rana et al. (2014). The thermal instability of a nanofluid saturating a rotating anisotropic porous medium has been studied by Agarwal et al. (2011). Recently Yadav and Kim (2014) studied the onset of transient Soret-driven buoyancy convection in nanoparticle suspensions with particle concentration–dependent viscosity in a porous medium.

Nanofluids are being looked upon as great coolants of the future, due to their enhanced thermal conductivities. Thus, studies need to be conducted involving nanofluids in porous media and without it. Due to the very small size of suspended nanoparticles, nanofluids form very stable colloidal systems with very little settling; a significant enhancement of effective thermal conductivity in comparison with the base fluid is observed. Various possible explanations for the heat transfer have been suggested and applied to standard problems such as onset of convection in a horizontal layer uniformly heated from below. When the layer is a porous medium this problem is commonly known as the Horton-Rogers-Lapwood (HRL) problem.

One approach is to follow Buongiorno (2006) who, after considering alternative agencies, proposed a model incorporating the effects of Brownian diffusion and the thermophoresis. This model was applied to the HRL problem by Nield and Kuznetsov (2011) and Kuznetsov and Nield (2011). Both Brownian diffusion and thermophoresis give rise to cross-diffusion terms that are in some ways analogs to the familiar Soret and Dufour cross-diffusion terms that arise with a binary fluid. In this model, fluid properties such as thermal conductivity and viscosity are given specified uniform values.

An alternative approach is to ignore special phenomena such as Brownian motion and thermophoresis but instead examine the effects of variation of thermal conductivity and viscosity with nanoparticle fluid fraction, utilizing expressions obtained using the theory of mixtures. This approach was employed by Tiwari and Das (2007) and is followed in this article in combination with the cross-diffusion effects. It is assumed that the nanofluid is diluted so that the volume fraction is small compared with unity, then the basic solution is such that this volume fraction is a linear function of the vertical coordinate. Thus, to a first approximation, the thermal conductivity and the viscosity can be taken as weak functions of the vertical coordinate. This means that we can treat the problem as one involving a weakly heterogeneous porous medium using an approach developed by Nield (2008) to obtain an approximate analytical solution. Nield and Kuznetsov (2012) concluded that the variation of viscosity and thermal conductivity enter an expression for the stability boundaries and the consequence of these factors was to increase the critical value of the Rayleigh number. The objective of the present analysis is to include the individual effects of the governing parameters on the stationary and oscillatory linear stability for Nield and Kuznetsov (2012). Further, the nonlinear steady and unsteady stability analysis is studied in terms of concentration and thermal Nusselt numbers.

2. ANALYSIS

2.1 Conservation Equations

We select a coordinate frame in which the z-axis is aligned vertically upwards. We consider a horizontal layer of fluid confined between the planes \( z^* = 0 \) and \( z^* = H \). Asterisks are used to denote dimensional variables. Each boundary wall is assumed to be perfectly thermally conducting. The temperatures at the lower and upper boundary are taken to be \( T^*_0 + \Delta T^* \) and \( T^* \). The Oberbeck Boussinesq approximation is employed. In the linear stability theory being applied here, the temperature change in the fluid is assumed to be small in comparison with \( T^*_0 \); the conservation equation takes the form

\[
\nabla^* \cdot \mathbf{v}^*_D = 0
\]

(1)

Here, \( \mathbf{v}^*_D \) is the nanofluid Darcy velocity. We write \( \mathbf{v}^*_D = (u^*, v^*, w^*) \).

In the presence of thermophoresis, the conservation equation for the nanoparticles, in the absence of chemical reactions, takes the form

\[
\frac{\partial \phi^*}{\partial t^*} + \nabla \cdot \mathbf{v}^*_D = \nabla^* \cdot \left[ D_B \nabla \phi^* + D_T \frac{\nabla^* T^*}{T^*} \right], \tag{2}
\]

where \( \phi^* \) is the nanoparticle volume fraction, \( \varepsilon \) is the porosity, \( T^* \) is the temperature, \( D_B \) is the Brownian diffusion coefficient, and \( D_T \) is the thermophoretic diffusion coefficient.

If one introduces a buoyancy force and adopts the Boussinesq approximation, and uses the Darcy model for a porous medium, then the momentum equation can be written as
Here $\rho$ is the overall density of the nanofluid, which we now assume to be given by
\[ \rho = \Phi^* \rho_p + (1 - \Phi^*) \rho_0 \left[ 1 - \beta_T (T^* - T_0^*) \right], \tag{4} \]
where $\rho_p$ is the particle density, $\rho_0$ is a reference density for the fluid, and $\beta_T$ is the thermal volumetric expansion. The thermal energy equation for a nanofluid can be written as
\[ (pc)_m \frac{\partial T^*}{\partial t} + (pc)_f \nabla^* \cdot \nabla^* T^* = k_m \nabla^{*2} T^* + \epsilon (pc)_p \times \left[ D_B \nabla^* \Phi^* \cdot \nabla^* + D_T \frac{\nabla^{*2} T^*}{T_0^*} \right]. \tag{5} \]
The conservation of nanoparticle mass requires that
\[ \frac{\partial \Phi^*}{\partial t^*} + \frac{1}{\epsilon} \nabla^* \cdot \nabla^* \Phi^* = D_B \nabla^{*2} \Phi^* + \frac{D_T}{T_0^*} \nabla^{*2} T^*. \tag{6} \]
Here $c$ is the fluid specific heat (at constant pressure), $k_m$ is the overall thermal conductivity of the porous medium saturated by the nanofluid, and $c_p$ is the nanoparticle specific heat of the material constituting the nanoparticles.

Thus,
\[ k_m = \epsilon k_{eff} + (1 - \epsilon) k_s, \tag{7} \]
where $\epsilon$ is the porosity, $k_{eff}$ is the effective conductivity of the nanofluid (fluid plus nanoparticles), and $k_s$ is the conductivity of the solid material forming the matrix of the porous medium.

We now introduce the viscosity and the conductivity dependence on nanoparticle fraction. Following Tiwari and Das (2007), we adopt the formulas, based on a theory of mixtures,
\[ \frac{\mu_{eff}}{\mu_f} = \frac{1}{(1 - \Phi^*)^{2.5}}, \tag{8} \]
\[ \frac{k_{eff}}{k_f} = \frac{(k_p + 2k_f) - 2\Phi^*(k_f - k_p)}{(k_p + 2k_f) + \Phi^*(k_f - k_p)}, \tag{9} \]
where $k_f$ and $k_p$ are the thermal conductivities of the fluid and the nanoparticles, respectively.

Equation (8) was obtained by Brinkman (1952), and Eq. (9) is the Maxwell-Garnett formula for a suspension of spherical particles that dates back to Maxwell (1904).

In the case where $\Phi^*$ is small compared with unity, we can approximate these formulas by
\[ \frac{\mu_{eff}}{\mu_f} = 1 + 2.5\Phi^*. \]
\[ \frac{k_{eff}}{k_f} = \frac{(k_p + 2k_f) - 2\Phi^*(k_f - k_p)}{(k_p + 2k_f) + \Phi^*(k_f - k_p)} = 1 + 3\Phi^* \frac{(k_p - k_f)}{(k_p + 2k_f)}. \tag{10} \]

We assume that the temperature and the volumetric fraction of the nanoparticles are constant on the boundaries. Thus the boundaries conditions are
\[ w^* = 0, \quad T^* = T_0^* + \Delta T^*, \quad \Phi^* = \Phi_0^* \text{ at } z^* = 0, \]
\[ w^* = 0, \quad T^* = T_0^* + \Delta T^*, \quad \Phi^* = \Phi_1^* \text{ at } z^* = H. \tag{11} \]

We introduce dimensionless variables as follows. We define
\[ (x, y, z) = (x^*, y^*, z^*)/H, \quad t = t^* \alpha_m/\sigma H^2, \]
\[ (u, v, w) = (u^*, v^*, w^*)H/\alpha_m, \quad p = p^*K/\mu_f \alpha_m, \tag{12} \]
where
\[ \alpha_m = \frac{k_m}{(pc_p)_f}, \quad \sigma = \frac{(pc_p)_m}{(pc_p)_f}, \quad m = \frac{\mu_{eff}}{\mu_f}. \tag{13} \]

From Eqs. (7), (10), and (13), we have
\[ m = 1 + 2.5 \left[ \Phi_0^* + \Phi^* (\Phi_1^* - \Phi_0^*) \right], \]
\[ k^* = \frac{k_p}{k_f}, \quad \tilde{k}^* = \frac{k_s}{k_f}, \quad \tilde{k} = \frac{k_m}{k_f}. \tag{14} \]
Then Eqs. (1) and (3) with Eqs. (4), (5), (2), (12), (13) take the form
\[ \nabla \cdot v = 0, \tag{15} \]
\[ 0 = -\nabla p - m \nu - Ra_M \dot{\theta}_z + Ra_T \dot{\theta}_z - Ra_N \Phi \dot{\theta}_z, \tag{16} \]
\[ \frac{\partial T}{\partial t} + v \cdot \nabla T = k \nabla^{2} T + \frac{N_B}{Le} \nabla \Phi \cdot \nabla T + \frac{N_A}{Le} \nabla \cdot \nabla T, \tag{17} \]
\[ \frac{1}{\sigma^*} \frac{\partial \Phi}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla \Phi = \frac{1}{Le} \nabla^{2} \Phi + \frac{N_A}{Le} \nabla^{2} T, \tag{18} \]
According to Buongiorno (2006), for most nanofluids in-
Here Le = \( \alpha \), \( V \) olume 6, Number 1, 2015

2.2 Basic Solution

We seek a time-independent quiescent solution of
Eqs. (15)–(19) with temperature and nanoparticle volume
fraction varying in the \( z \)-direction only, that is a solution of
the form

\[
v = 0, \quad p = p_b(z), \quad T = T_b(z), \quad \varphi = \varphi_b(z).
\] (20)

Equations (16)–(18) reduce to

\[
0 = -\frac{dp_b}{dz} - Ra_M + Ra_T T_b - Ra_N \varphi_b,
\] (21)

\[
k_\epsilon \frac{d^2 T_b}{dz^2} + \frac{N_B \varphi_b}{Le} \frac{d\varphi_b}{dz} + \frac{N_A N_B}{Le} \left( \frac{dT_b}{dz} \right)^2 = 0,
\] (22)

\[
\frac{d^2 \varphi_b}{dz^2} + Na_k \frac{d^2 T_b}{dz^2} = 0.
\] (23)

According to Buongiorno (2006), for most nanofluids
investigated so far \( Le/(\varphi_1 - \varphi_0) \) is large, of order \( 10^3 \)
and since the nanoparticle fraction decrement is typically
no smaller than \( 10^3 \) so this means that \( Le \) is large,
of order \( 10^2 \) to \( 10^3 \), while \( N_A \) is no greater than about 10. Using this approximation, the basic solution is found to be

\[
T_b = 1 - z,
\] (24a)

and so

\[
\varphi_b = z.
\] (24b)

2.3 Perturbation Solution

We now superimpose perturbations on the basic solution. We write

\[
v = v' + p_b + p', \quad T = T_b + T', \quad \varphi = \varphi_b + \varphi'.
\] (25)

Substitute in Eqs. (14)–(19), and linearize by neglecting
products of primed quantities. The following equations are obtained when Eq. (24) is used:

\[
\nabla \cdot v' = 0,
\] (26)

\[
0 = -\nabla p' - m v' + Ra_T' \hat{e}_z - Ra_N \varphi' \hat{e}_z,
\] (27)

\[
\frac{\partial T'}{\partial t} - w' = \nabla^2 T' + \frac{N_B}{Le} \left( \frac{\partial T'}{\partial z} - \frac{\partial \varphi'}{\partial z} \right)
\] (28)

\[
- \frac{2N_A N_B}{Le} \frac{\partial T'}{\partial z},
\]

\[
\frac{1}{\sigma} \frac{\partial \varphi'}{\partial t} + \frac{1}{\varepsilon} w' = \frac{1}{Le} \nabla^2 \varphi' + \frac{N_A}{Le} \nabla^2 T',
\] (29)

\[
w' = 0, \quad T' = 0, \quad \varphi' = 0, \quad at \quad z = 0, \quad and \quad at \quad z = 1,
\] (30)

where now we can approximate the viscosity and conductivity distributions by substituting the basic solution expression for \( \varphi \), namely that given by Eq. (24), into Eqs. (14)–(19); we obtain

\[
m(z) = 1 + 2.5 \left[ \varphi_0^* + \varphi(\varphi_1^* - \varphi_0^*) \right] z,
\] (31a)

\[
k(z) = \varepsilon \left( 1 + 3[\varphi_0^* + \varphi(\varphi_1^* - \varphi_0^*) z] \frac{k_p - 1}{k_p + 2} \right)
\] (31b)

It will be noted that the parameter \( Ra_M \) is just a measure
of the basic static pressure gradient and is not involved
in these and subsequent equations.

We now recognize that we have a situation where properties
are heterogeneous. These are now the viscosity and conductivity (rather than the more usual ones, namely permeability and conductivity) and we can now proceed as in a number of papers by the present authors that are surveyed by Nield (2008). We assume that the heterogeneity is weak in the sense that the maximum variation of a property over the domain considered is small compared with the mean value of that property.

The six unknowns \( u', \nu', \epsilon', \varphi', T', \varphi' \) can be reduced
to three by operating on Eq. (27) with \( \hat{e}_z \) curl curl and using the identity curl curl \( \nabla \times \nabla \times \) together with Eq. (26) and the weak heterogeneity approximation. The result [after using Eq. (31)] is

\[
m(z)\nabla^2 w' = Ra_T \nabla^2 H - Ra_N \nabla^2 H \varphi'.
\] (32)
$\nabla^2_H$ is the two-dimensional Laplacian operator on the horizontal plane.

The differential Eqs. (27), (28), (32) and the boundary conditions (30) constitute a linear boundary-value problem, that can be solved using the method of normal modes. We write

$$w', T', \phi' = [W(z), \Theta(z), \Phi(z)] \times \exp(st + ilx + imy), \quad (33)$$

and substitute into the differential equations to obtain

$$m(z) \left(D^2 - \alpha^2\right) W + Ra_T \alpha^2 \Theta - Ra_N \alpha^2 \Phi = 0, \quad (34)$$

$$W + k(z) \left(D^2 + \frac{N_B}{Le} D - \frac{2N_A \cdot N_B}{Le} D - s\right) \Theta - \frac{N_B}{Le} D \Phi = 0, \quad (35)$$

$$\frac{1}{\epsilon} W - \frac{NA}{Le} \left(D^2 - \alpha^2\right) \Theta - \frac{1}{Le} \left(D^2 - \alpha^2\right) - \frac{1}{\sigma} s \Phi = 0, \quad (36)$$

$$W = 0, \quad \Theta = 0, \quad \Phi = 0 \text{ at } z = 0, \quad \text{and } z = 1, \quad (37)$$

where

$$D \equiv \frac{d}{dz} \text{ and } \alpha^2 = (l^2 + m^2)^{1/2}. \quad (38)$$

Thus $\alpha$ is a dimensionless horizontal wave number.

For neutral stability the real part of $s$ is zero. Hence we now write $s = i\omega$, where $\omega$ is real and is a dimensionless frequency.

We now employ a Galerkin-type weighted residuals method to obtain an approximate solution to the system of (34)–(37). We choose as trial functions (satisfying the boundary conditions)

$$W_p, \Theta_p, \Phi_p; \quad p = 1, 2, 3 \ldots$$

and write

$$W = \sum_{p=1}^{N} A_p W_p, \quad \Theta = \sum_{p=1}^{N} B_p \Theta_p, \quad \Phi = \sum_{p=1}^{N} C_p \Phi_p. \quad (39)$$

Substitute into Eqs. (34)–(37), and make the expressions on the left-hand sides of those equations (the residuals) orthogonal to the trial functions, thereby obtaining a system of $3N$ linear algebraic equations in the $3N$ unknowns $A_p, B_p, C_p, p = 1, 2, \ldots, N$. The vanishing of the determinant of coefficients produces the eigenvalue equation for the system. One can regard $Ra_T$ as the eigenvalue. Thus $Ra_T$ is found in terms of the other parameters as

$$Ra_T = \frac{1}{\left(\frac{J}{Le} + \frac{s}{\sigma}\right)} \left[mJ \left(\frac{J}{Le} + \frac{s}{\sigma}\right) (Jk + s) - \frac{Ra_N \alpha^2}{\epsilon} (kJ + s) - Ra_N \alpha^2 \frac{NA}{Le} J\right],$$

where for shorthand we have written $J = (\pi^2 + \alpha^2)$.

3. LINEAR STABILITY ANALYSIS

If all initial states are classified as stable or unstable, according to the criteria stated, then the locus which separates the two classes of states defines the states of marginal stability of the system. By this definition, a marginal state is a state of neutral stability. Marginal stability (neutral stability) states can be classified into two classes, viz., nonoscillatory (stationary) and oscillatory convections.

3.1 Nonoscillatory Convection

For the validity of principle of exchange of stabilities (i.e., steady case), we have $s = 0$ (i.e., $s = s_r + is_i = s_r = s_i = 0$) at the margin of stability.

For the homogeneous case ($m = 1, k = 1$), the nonoscillatory stability boundary ($s = 0$ in the $Ra_T$ expression) is given, using the one-term Galerkin approximation, by

$$Ra_T = \frac{\left(\pi^2 + \alpha^2\right)^2}{Na^2 + \frac{Le}{\epsilon}} \cdot (40)$$

In the present case, where viscosity and conductivity variations are incorporated, the critical wave number is unchanged and the stability boundary becomes

$$Ra_T = \frac{\left(\pi^2 + \alpha^2\right)^2 km}{\alpha^2} - Ra_N \left(N_A + \frac{Le}{\epsilon}\right). \quad (41)$$

We observe the effect of viscosity variation is to increase the critical Rayleigh number by a factor $m$, and also the additional effect of conductivity variation is to increase the critical Rayleigh number by a factor $k$.

3.2 Oscillatory Convection

For the case of oscillatory convection, we have $s = i\omega$, where $\omega$ is a real quantity, (i.e., $s = s_r + is_i$, $s_r = 0, s_i = \omega$) at the margin of stability. Hence, we
now consider the case \( s = i \omega \). We confine ourselves to
the one-term Galerkin approximation. The oscillatory sta-
bility boundary is found to be given by

\[
\text{Ra}_T = \frac{1}{\left( \frac{J}{\text{Le}} + \frac{i \omega}{\sigma} \right) \alpha^2} \left[ m J \left( \frac{J}{\text{Le}} + \frac{i \omega}{\sigma} \right) (Jk + i \omega) - \frac{\text{Ra}_N \alpha^2}{\varepsilon} (Jk + i \omega) - \frac{\text{Ra}_N \alpha^2 N_A}{\text{Le}} J \right].
\]

For the homogeneous case \((m = 1, k = 1)\), the oscillatory convection becomes

\[
\text{Ra}_T = \frac{1}{\left( \frac{J}{\text{Le}} + \frac{i \omega}{\sigma} \right) \alpha^2} \left[ J \left( \frac{J}{\text{Le}} + \frac{i \omega}{\sigma} \right) (J + i \omega) - \frac{\text{Ra}_N \alpha^2}{\varepsilon} (J + i \omega) - \frac{\text{Ra}_N \alpha^2 N_A}{\text{Le}} J \right].
\]

4. NONLINEAR STABILITY ANALYSIS

For simplicity, we consider the case of two-dimensional
rolls, assuming all physical quantities to be independent of \( y \). Eliminating the pressure and introducing the stream function we obtain

\[
m \nabla^2 \psi + \text{Ra}_T \frac{\partial T}{\partial x} - \text{Ra}_N \frac{\partial S}{\partial x} = 0, \tag{42}
\]

\[
\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial x} = k \nabla^2 T + \frac{\partial (\Psi, T)}{\partial (x, z)}, \tag{43}
\]

\[
\frac{1}{\sigma} \frac{\partial \psi}{\partial t} + \frac{1}{\varepsilon} \frac{\partial ^2 \psi}{\partial x^2} = \frac{1}{\text{Le}} \nabla^2 S + \frac{N_A}{\text{Le}} \nabla^2 T + \frac{1}{\varepsilon} \frac{\partial (\Psi, S)}{\partial (x, z)}. \tag{44}
\]

We solve Eqs. (42)–(44) subjecting them to stress-free, isothermal, iso-nanoconcentration boundary conditions:

\[
\psi = \frac{\partial^2 \psi}{\partial z^2} = T = \Phi = 0 \quad \text{at} \quad z = 0, 1. \tag{45}
\]

To perform a local nonlinear stability analysis, we take the following Fourier expressions:

\[
\begin{align*}
\psi &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn}(t) \sin(m \alpha x) \sin(n \pi z) , \\
T &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn}(t) \cos(m \alpha x) \sin(n \pi z) , \\
S &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn}(t) \cos(m \alpha x) \sin(n \pi z).
\end{align*}
\]

Further, we take the modes \((1, 1)\) for stream function, and
\((0, 2)\) and \((1, 1)\) for temperature, and nanoparticle concentration, to get

\[
\begin{align*}
\psi &= A_1(t) \sin(\alpha x) \sin(\pi z), \\
T &= A_2(t) \cos(\alpha x) \sin(2\pi z) + A_3(t) \sin(2\pi z), \\
S &= A_4(t) \cos(\alpha x) \sin(\pi z) + A_5(t) \sin(2\pi z),
\end{align*}
\]

where the amplitudes \(A_1(t), A_2(t), A_3(t), A_4(t), A_5(t)\)
are functions of time and are to be determined.

Taking the orthogonality condition with the eigenfunc-
tions associated with the considered minimal model, we get

\[
\begin{align*}
\frac{dA_1}{dt} &= \frac{\alpha}{m \delta^2} \left[ \text{Ra}_N A_4(t) - \text{Ra}_T A_2(t) \right], \\
\frac{dA_2}{dt} &= -\left[ \alpha A_1(t) + k \delta^2 A_2(t) + \alpha \pi A_1(t) A_3(t) \right], \\
\frac{dA_3}{dt} &= -k 4 \pi^2 A_3(t) + \frac{\alpha \pi}{2} A_1(t) A_2(t), \\
\frac{dA_4}{dt} &= -\sigma \left[ \frac{1}{\varepsilon} \alpha A_1(t) + \delta^2 \left( \frac{A_4(t)}{\text{Le}} + \frac{N_A}{\text{Le}} A_2(t) \right) + \frac{1}{\varepsilon} \alpha A_1(t) A_5(t) \right], \\
\frac{dA_5}{dt} &= -\sigma \left[ \frac{1}{\varepsilon} 4 \pi^2 A_5(t) + \frac{1}{2 \varepsilon} A_1(t) A_4(t) \right].
\end{align*}
\]

In case of steady motion \(dA_i/dt = 0\) and write all \(A_i\) in terms of \(A_1\).

Thus we get

\[
\begin{align*}
A_1(t) &= \frac{\alpha}{m \delta^2} \left[ \text{Ra}_N A_4(t) - \text{Ra}_T A_2(t) \right], \\
D_2 &= -\left[ \alpha A_1(t) + k \delta^2 A_2(t) + \alpha \pi A_1(t) A_3(t) \right], \\
D_3 &= -k 4 \pi^2 A_3(t) + \frac{\alpha \pi}{2} A_1(t) A_2(t), \\
D_4 &= -\sigma \left[ \frac{1}{\varepsilon} \alpha A_1(t) + \delta^2 \left( \frac{A_4(t)}{\text{Le}} + \frac{N_A}{\text{Le}} A_2(t) \right) + \frac{1}{\varepsilon} \alpha A_1(t) A_5(t) \right], \\
D_5 &= -\sigma \left[ \frac{1}{2 \varepsilon} \alpha A_1(t) A_4(t) \right], \\
\end{align*}
\]

and \(D_2 = D_3 = D_4 = D_5 = 0\).

The above system of simultaneous autonomous ordinary
differential equations is solved numerically using the
Runge-Kutta-Gill method. One may also conclude that the trajectories of the above equations will be confined to the finiteness of the ellipsoid. Thus, the effect of the parameters $Ra_N, Le, N_A$ on the trajectories is to attract them to a set of measure zero, or to a fixed point.

5. HEAT AND NANOPARTICLE CONCENTRATION TRANSPORT

The thermal Nusselt number, $Nu$ is defined as

$$ Nu(t) = \frac{\text{Heat transport by (conduction+convection)}}{\text{Heat transport by conduction}} = 1 + \left[ \frac{2\pi}{\alpha} \int_0^\infty \frac{\partial T}{\partial z} dx \right]_{z=0} $$

Substituting expressions (24) and (47) in the above equation we get

$$ Nu(t) = 1 - 2\pi A_3(t). $$

The nanoparticle concentration Nusselt number (Sherwood number), $Sh$ is defined similar to the thermal Nusselt number. Following the procedure adopted for arriving at $Nu(t)$, one can obtain the expression for $Sh(t)$ in the form

$$ Sh(t) = \{ [1 - 2\pi A_5(t)] + N_A [1 - 2\pi A_3(t)] \} $$

6. RESULTS AND DISCUSSION

The onset of convection in a porous medium saturated by a nanofluid is investigated using a linear and nonlinear theory. The linear theory gives the condition for the onset of stationary and oscillatory convections. The expressions of thermal Rayleigh number for stationary and oscillatory convections are given by Eqs. (41) and (42), respectively.

Figure 1(a)–1(f) shows the effect of various parameters on the neutral stability curves for stationary convection with variation in one of these parameters. In all these plots, it is interesting to note that the value of $Ra_T$ starts from a higher note, falls rapidly with increasing $\alpha$, and finally increases steadily. The effect of nanoparticle concentration Rayleigh number $Ra_N$ is shown in Fig. 1(a). It is shown that the thermal Rayleigh number decreases with increase in nanoparticle concentration Rayleigh number $Ra_N$, which means that nanoparticle concentration Rayleigh number $Ra_N$ destabilizes the system. It should be noted that the negative value of $Ra_N$ indicates a bottom-heavy case, while a positive value indicates a top-heavy case. The stationary convection is possible for both top- and bottom-heavy nanoparticles. It can be seen from Figs. 1(a1) and 1(a2) that positive values of $Ra_N$ will destabilize whereas negative values of $Ra_N$ will stabilize the system for stationary convection. The effect of Lewis number $Le$ on the thermal Rayleigh number is shown in Fig. 1(b1). One can see that the thermal Rayleigh number increases with increase in Lewis number, indicating that the Lewis number stabilizes the system. However, when a top-heavy nanoparticle is consid-

FIG. 1
FIG. 1: Neutral curves on stationary convection for different values of (a) nanoparticle concentration Rayleigh number $\text{Ra}_N$ [(a$_1$) top-heavy $\text{Ra}_N$, (a$_2$) bottom-heavy $\text{Ra}_N$], (b) Lewis number [(b$_1$) bottom-heavy $\text{Ra}_N$, (b$_2$) top-heavy $\text{Ra}_N$ (a$_2$)]; (c) modified diffusivity ratio $N_A$; (d) porosity $\varepsilon$; (e) viscosity ratio $m$; (f) conductivity ratio $k$
erated then the effect of the Lewis number shows a destabilizing effect [Fig. 1(b)]. The effect of modified diffusivity ratio $N_A$ on the thermal Rayleigh number is shown in Fig. 1(c). It is shown in Fig. 1(c) that as $N_A$ increases $Ra_T$ increases and hence $N_A$ has a stabilizing effect on the system. From Fig. 1(d), one can observe that as porosity $\varepsilon$ increases, thermal Rayleigh number decreases which means that the porosity advances the onset of convection. The effect of viscosity ratio $m$ and conductivity ratio $k$ on the thermal Rayleigh number is depicted in Figs. 1(e) and 1(f) respectively; these figures show that as $m$ and $k$ increases, $Ra_T$ increases which indicates that $m$ and $k$ will stabilize the system. The effect of concentration Rayleigh number $Ra_N$ and Lewis number $Le$ on thermal Rayleigh number $Ra_T$ for stationary convection shows results similar to those obtained by Sheu (2011). Nield and Kuznetsov (2009) mentioned that $Ra_N$ is defined in a way so that it is positive when the applied particle density increases upwards (the destabilizing situation). It is also noted that $Ra_T$ takes a negative value when $Ra_N$ is sufficiently large. In this case the destabilizing effect of concentration is so great that the bottom of the fluid layer must be cooled relative to the top in order to produce a state of neutral stability. Therefore positive values of $Ra_N$ destabilize the system in stationary convection.

Figure 2(a)–2(g) displays the variation of thermal Rayleigh number for oscillatory convection with respect to various parameters. In Fig. 2(a) it is seen that for negative values of $Ra_N$ (bottom-heavy case) the thermal Rayleigh number decreases as $Ra_N$ increases which will...
Convective Transport in a Nanofluid Saturated Porous Layer

FIG. 2: Neutral curves on oscillatory convection for different values of (a) nanoparticle concentration Rayleigh number $Ra_N$, (b) Lewis number $Le$, (c) modified diffusivity ratio $N_A$, (d) porosity $\varepsilon$, (e) thermal capacity ratio $\sigma$, (f) viscosity ratio $m$, (g) conductivity ratio $k$

advance the onset of convection. As the Lewis number $Le$ increases, the thermal Rayleigh number $Ra_T$ decreases, as seen in Fig. 2(b), which implies that the Lewis number $Le$ destabilizes the system. The modified diffusivity ratio $N_A$ does not show any effect on the oscillatory convection [Fig. 2(c)]. From Fig. 2(d), one can reveal that the porosity $\varepsilon$ destabilizes the system for oscillatory convection which is an increase in $\varepsilon$ that decreases the thermal Rayleigh number. As the thermal capacity ratio $\sigma$ increases, the thermal Rayleigh number also increases as can be observed in Fig. 2(e), which implies that $\sigma$ has a stabilizing effect on the system for oscillatory convection. The effect of viscosity ratio $m$ and conductivity ratio $k$ on thermal Rayleigh number is depicted in Figs. 2(f) and 2(g), respectively. From these figures one can conclude that both $m$ and $k$ increase the thermal Rayleigh number for oscillatory convection, thus delaying the onset of convection. The effect of concentration Rayleigh number $Ra_N$, Lewis number $Le$, and thermal capacity ratio $\sigma$ on thermal Rayleigh number $Ra_T$ for oscillatory convection show results similar to those obtained by Sheu (2011).

The nonlinear analysis provides not only the onset threshold of finite amplitude motion but also the information of heat and mass transports in terms of Nusselt Nu and Sherwood Sh numbers. The Nu and Sh are computed as the functions of $Ra_T$, and the variations of these nondi-
dimensional numbers with $Ra_{T}$ for different parameter values are depicted in Figs. 3(a)–3(e) and 4(a)–4(e), respectively. In Figs. 3(a)–3(e) and 4(a)–4(e) it is observed that in each case, the Sherwood number is always greater than the Nusselt number and both Nusselt number and Sherwood number start with the conduction state value 1 at the point of onset of steady finite amplitude convection. When $Ra_{T}$ is increased beyond $Ra_{Tc}$, there is a sharp in-

FIG. 3: Variation of Nusselt number $Nu$ with critical Rayleigh number for different values of (a) nanoparticle concentration Rayleigh number $Ra_{N}$, (b) Lewis number $Le$, (c) modified diffusivity ratio $N_{A}$, (d) viscosity ratio $m$, (e) conductivity ratio $k$
FIG. 4: Variation of Sherwood number $Sh$ with critical Rayleigh number for different values of (a) nanoparticle concentration Rayleigh number $Ra_N$, (b) Lewis number $Le$, (c) modified diffusivity ratio $N_A$, (d) viscosity ratio $m$, (e) conductivity ratio $k$.
crease in the values of both \(Nu\) and \(Sh\). However, further increase in \(Ra_N\) will not change \(Nu\) and \(Sh\) significantly. It is to be noted that the upper bound of \(Nu\) is 3 (similar results were obtained by Malasheety et al., 2011). It should also be noted that the upper bound of \(Sh\) is not 3 (similar results were obtained by Bhadauria and Agarwal, 2011). The upper bound of \(Nu\) remains 3 only for both clear and nanofluid, whereas, the upper bound for \(Sh\) for clear fluid is 3 but for nanofluid it is not fixed.

In Figs. 3(a) and 4(a) we observe that as the concentration Rayleigh number \(Ra_N\) increases, the value of \(Nu\) and \(Sh\) also increases, thus showing an increase in the rate of heat and mass transport. Figures 3(b) and 4(b) shows that as the Lewis number increases both \(Nu\) and \(Sh\) decrease, which implies that increasing the Lewis number suppresses the heat and mass transport. In Figs. 3(c) and 4(c) we observe that on increasing modified diffusivity ratio \(N_A\) there is no effect on the Nusselt number, whereas it increases the Sherwood number (which is similar to the result observed by Bhadauria and Agarwal, 2011). As the viscosity ratio \(m\) [Figs. 3(d) and 4(d)] and conductivity ratio \(k\) [Figs. 3(e) and 4(e)] increase both the \(Nu\) and \(Sh\) decrease, implying that \(m\) and \(k\) suppress the heat and mass transports.

The linear solutions exhibit a considerable variety of behavior of the system, and the transition from linear to nonlinear convection can be quite complicated, but interesting to deal with. There is a need to study time-dependent results to analyze the same. The transition can be well understood by the analysis of Eq. (48) whose solution gives a detailed description of the two-dimensional problem. The autonomous system of unsteady finite amplitude equations is solved numerically using the Runge-Kutta method. The Nusselt and Sherwood numbers are evaluated as the functions of time \(t\); the unsteady transient behavior of \(Nu\) and \(Sh\) is shown graphically in Figs. 5(a)–5(e) and 6(a)–6(e), respectively.

These figures indicate that initially when time is small, there occur large scale oscillations in the values of \(Nu\) and \(Sh\) indicating an unsteady rate of heat and mass transport in the fluid. As time passes, these values approach their steady state values corresponding to a near convection stage.

Figure 5(a) depicts the transient nature of the Nusselt number on nanoparticle concentration Rayleigh number \(Ra_N\). It is observed that as \(Ra_N\) increases \(Nu\) decreases, thus showing a decrease in the heat transport, which is similar to the result observed by Agarwal et al. (2012). From Figs. 5(b), 5(d), and 5(e), we observe that as the Lewis number, viscosity ratio, and conductivity ratio increase \(Nu\) decreases, indicating that there is retardation on heat transport. The modified diffusivity ratio enhances the heat transport as seen in Fig. 5(c).

It is seen from Figs. 6(a), 6(d), and 6(e) that as nanoparticle concentration Rayleigh number \(Ra_N\), viscosity ratio \(m\), and conductivity ratio \(k\) increase the Sherwood number (concentration Nusselt number) decreases, which implies the suppression of mass transport. The mass transport is enhanced for Lewis number \(Le\) and modified diffusivity ratio \(N_A\) as seen in Figs. 6(b) and 6(c), respectively.

7. CONCLUSIONS

We considered linear stability analysis in a horizontal porous medium saturated by a nanofluid, heated from below and cooled from above, using the Darcy model which incorporates the effect of Brownian motion along with thermophoresis. Further, the viscosity and conductivity dependence on nanoparticle fraction was also adopted following Tiwari and Das (2007). Linear analysis has been made using normal mode technique. However, for weakly nonlinear analysis, truncated Fourier series representation having only two terms is considered. We draw the following conclusions:

1. For stationary convection Lewis number \(Le\), modified diffusivity ratio \(N_A\), viscosity ratio \(m\), and conductivity ratio \(k\) have a stabilizing effect while porosity \(\varepsilon\) destabilizes the system for negative values of \(Ra_N\). The positive values of \(Ra_N\) will destabilize the system and negative values of \(Ra_N\) will stabilize the system for variations of \(Ra_N\). The Lewis number destabilizes the system for positive values of \(Ra_N\).

2. For oscillatory convection thermal capacity ratio \(\sigma\), viscosity ratio \(m\), and conductivity ratio \(k\) stabilize the system, whereas nanoparticle concentration Rayleigh number \(Ra_N\), Lewis number \(Le\), and porosity \(\varepsilon\) destabilize the system.

3. For steady finite amplitude motions, the nanoparticle concentration Rayleigh number \(Ra_N\) enhances the heat and mass transports while the heat and mass transport decreases with increase in the values of Lewis number \(Le\), viscosity ratio \(m\), and conductivity ratio \(k\). The mass transport increases with increase in modified diffusivity ratio \(N_A\).

4. The transient Nusselt number and Sherwood number increase with increase in Lewis number \(Le\) and
FIG. 5: Transient Nusselt number $Nu$ with time for different values of (a) nanoparticle concentration Rayleigh number $Ra_N$, (b) Lewis number $Le$, (c) modified diffusivity ratio $N_A$, (d) viscosity ratio $m$, (e) conductivity ratio $k$
FIG. 6: Transient Sherwood number $Sh$ with time for different values of (a) nanoparticle concentration Rayleigh number $Ra_N$, (b) Lewis number $Le$, (c) modified diffusivity ratio $N_A$, (d) viscosity ratio $m$, (e) conductivity ratio $k$.
modified diffusivity ratio $N_A$, and decrease with nanoparticle concentration Rayleigh number $Ra_N$, viscosity ratio $m$, and conductivity ratio $k$.

5. The effect of time on the transient Nusselt number and Sherwood number is found to be oscillatory when $t$ is small. However, when $t$ becomes very large both the transient Nusselt and Sherwood value approach their steady state values.

REFERENCES


