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DEEP BED FILTRATION WITH TIME-DEPENDENT INPUT CONDITIONS

Ali J. Chamkha¹* & Jasem Al-Humoud²

¹Mechanical Engineering Department, Prince Mohammad Bin Fahad University, P.O. Box 1664, Al-Khobar 31952, Kingdom of Saudi Arabia
²Civil Engineering Department, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

*Address all correspondence to: Ali J. Chamkha, E-mail: achamkha@yahoo.com

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This work focuses on the numerical solution of a one-dimensional continuum filtration model with time-dependent filter inlet concentration. Two different types of filter inlet concentration sources are considered. A general mass transfer that characterizes particle attachment and detachment effects is considered. The general problem is solved numerically by an efficient implicit finite-difference method. Comparisons with exact analytical solutions for special cases of the filtration model are performed and found to be in excellent agreement. A parametric study of some physical parameters is conducted and the results are presented graphically to illustrate interesting features of the solutions. It is found that the time-dependent inlet concentration sources have significant changes in the solute concentration, specific deposit, and filter efficiency at all filtration time stages compared with their corresponding constant inlet concentration source cases.

KEY WORDS: deep bed filtration, time-dependent inlet concentration, numerical solution, finite-difference method

1. INTRODUCTION

Mathematical modeling of the filtration of particulate suspensions through the porous medium of a filter is an integral part of constructing a prediction model of any water purification system or commercial system for separating multiphase materials. In general, the composition of the particulate suspension varies over time during the operation of such systems (Kochmarskii, 1989). Nevertheless, it has been assumed in many well-known studies (Shekhtman, 1961; Mints, 1964; Kul’skii and Strokach, 1986; Venitsianov and Rubinshtein, 1983) that the inlet concentration of impurities in filtration is constant. Mathematically speaking, the assumption of variable inlet concentration of impurities adds new degrees of freedom characterizing the filtration process, such as the frequency and amplitude of changes in impurity concentration for oscillatory-type inlet solute concentration and the coefficient and exponent for exponential-type inlet concentration. It is of interest to know the degree to which deviations from the constant inlet solute concentration assumption affect predictions of filter performance and the relationship between the fluctuating and constant components of the above types of variable inlet concentration.

The popular class of attachment functions discussed by Tien (1989) is shown to lead to a formulation allowing a single solution to be representative of all possible variations of inlet concentrations, superficial velocities, and initial porosities. For the sake of illustration of the effects of variable inlet solute concentration, the linear “chromatographylike” attachment/detachment function [see, for instance, Tien (1989) or Adin and Rajagopalan (1989)] is considered. It should be pointed out that filtration models differ mainly in the forms of the attachment/detachment functions employed therein. The complexity of these functions is determined by the number of phenomena they attempt to describe. The simplest models are those involving pure attachment such as that...
These models can be modified to account for filter clogging in a number of ways. The papers listed in Table 2.1 of Tien (1989) are representative of this work. Soo and Radke (1986) and Soo et al. (1986) showed that it was possible to use such models to describe attachment by simultaneous straining and interception. Models allowing for both attachment and detachment (particle unloading or reentrainment of previously captured particles) have been developed. Representative of this work are the papers by Adin (1978), Adin and Rebhun (1987), and Adin and Rajagopalan (1989). As discussed in a latter paper and by Rajagopalan and Chu (1982), it is possible to reinterpret such models to describe the reduction in attachment rate associated with the covering of filter grain surfaces. Privman et al. (1991) dealt with this phenomenon explicitly. Song and Elimelich (1993) extended this work by introducing an internal variable which describes the time evolution of the surface covering. It is interesting to note that all these models except the one associated with pure attachment predict s-shaped breakthrough curves which are qualitatively similar. The objective of the present work is to consider a continuum filtration theory in which the solute concentration at the inlet of the filter is time dependent and to illustrate how this variation affects the filter performance in comparison with the case of constant inlet concentration of impurities.

2. PROBLEM FORMULATION

Consider a one-dimensional continuum filtration model with an arbitrary general mass transfer function $M^*$. The solute transport in the porous medium of the filter takes place from a time-dependent inlet solute concentration source, and the superficial velocity and diffusion coefficient are assumed to be constant. The governing equations are based on the filtrate and filter mass balances respectively described by:

$$\epsilon_i \frac{\partial c(z,t)}{\partial t} + u \frac{\partial c(z,t)}{\partial z} = D \frac{\partial^2 c(z,t)}{\partial z^2} + \lambda M^*(\epsilon, c) \quad (1)$$

$$\frac{\partial \epsilon(z,t)}{\partial t} = -\lambda M^*(\epsilon, c) \quad (2)$$

where $t$ is time, $z$ is the one-dimensional spatial coordinate ($0 \leq z \leq L$, where $L$ is the filter length), $c(z,t)$ is the solute concentration, $u$ is the filter superficial velocity, $D$ is the hydrodynamic diffusion or dispersion coefficient.

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representing the sum of both the effects of molecular diffusion \((D_e)\) and mechanical dispersion \((\alpha^*u)\) where \(\alpha^*\) is usually called the dispersivity, \(M^*(\varepsilon,c)\) is a general mass transfer function, \(\lambda\) is the inverse filtration length, \(\varepsilon(z,t)\) is the filter porosity, and \(\varepsilon_i\) is the initial filter porosity. It should be noted that the first term on the left-hand side of Eq. (1) represents the transient or accumulation effect. The second term on the left-hand side of the equation represents the advection or convection effect which is defined as the transport of solute by the mean velocity in the flow stream. The first term on the right side of Eq. (1) accounts for the dispersion or diffusion effect which is responsible for the spreading of the solute in the medium. The last term of Eq. (1) represents the filtration attachment and/or detachment mechanisms which may take place (depending on the nature and properties of the solute and the porous medium of the filter).

The initial and boundary conditions for this problem are

\[
\begin{align*}
    c(z,0) &= 0, \quad \varepsilon(z,0) = \varepsilon_i \\
    c(0,t) &= c_if(t), \quad \frac{\partial c(L,t)}{\partial z} = 0
\end{align*}
\]  

It is convenient to work with dimensionless equations. This can be accomplished by using

\[
\eta = \frac{z}{L}, \quad \tau = \frac{t}{T}, \quad C(\eta, \tau) = \frac{c(z,t)}{c_i}, \quad s = \varepsilon_i - \varepsilon, \quad M = \frac{M^*}{uc_i}, \quad S = \frac{s}{uc_i T/L}
\]  

where \(T\) is a characteristic quantity of process time and \(s\) is the filter specific deposit.

Substituting Eq. (4) into Eqs. (1)–(3) gives the following respective equations and conditions:

\[
\begin{align*}
    N_2 \frac{\partial C(\eta, \tau)}{\partial \tau} + \frac{\partial C(\eta, \tau)}{\partial \eta} &= N_3 \frac{\partial^2 C(\eta, \tau)}{\partial \eta^2} \\
    -N_1 M(S,C)
    \quad \frac{\partial S(\eta, \tau)}{\partial \tau} &= N_1 M(S,C) \quad (5) \\
    C(\eta, 0) &= 0, \quad S(\eta, 0) = 0 \quad (6) \\
    C(0, \tau) &= f(\tau), \quad \frac{\partial C(1, \tau)}{\partial \eta} = 0
\end{align*}
\]

where

\[
N_1 = \lambda L, \quad N_2 = \frac{\varepsilon_i L}{uT}, \quad N_3 = \frac{D}{uL}
\]  

are dimensionless numbers which characterize the filtration process. In particular, \(N_1\) measures the influence of attachment rate, \(N_2\) the influence of disturbance front propagation, and \(N_3\) the influence of diffusion.

The linear “chromatographylike” attachment/detachment model for the mass transfer function \(M(C,S)\) [see, for instance, Tien (1998) or Adin and Rajagopalan (1989)] is often used in filtration modeling since it allows for a variety of possible cases. For this reason, it is adopted in this work. It is given by

\[
M(S,C) = C - \frac{N_5 S}{N_1} \quad (9)
\]

where \(N_1\) is a dimensionless constant characterizing the attachment rate and \(N_5\) is a dimensionless parameter characterizing the detachment rate. During the filtration process, particles get collected (attached) on the filter media and as more particles collect on the filter media and collected particles, some already-collected particles unload (detach) from the filter. Here \(N_1\) and \(N_5\) characterize the particle attachment and detachment mechanisms, respectively.

Substituting Eq. (9) into Eqs. (5) and (6) and simplifying yields

\[
\begin{align*}
    N_2 \frac{\partial C(\eta, \tau)}{\partial \tau} + \frac{\partial C(\eta, \tau)}{\partial \eta} &= N_3 \frac{\partial^2 C(\eta, \tau)}{\partial \eta^2} - N_1 C(\eta, \tau) \\
    + N_5 S(\eta, \tau) \quad (10) \\
    \frac{\partial S(\eta, \tau)}{\partial \tau} &= N_1 C(\eta, \tau) - N_5 S(\eta, \tau) \quad (11)
\end{align*}
\]

In filtration modeling, in addition to the solute concentration and the filter specific deposit (or porosity), information about the time evolution of the filter efficiency is of great importance. The filter efficiency is defined as follows:

\[
E(\tau) = 1 - \frac{c(L,t)}{c_i} = 1 - C(1, \tau) \quad (12)
\]

Two different time-dependent inlet concentration sources will be considered. The first will be an exponentially decaying source with time and the second will be an oscillating source. These sources take the following forms:

\[
\begin{align*}
    f(\tau) &= 1 + \exp(-\beta \tau) \quad (13) \\
    f(\tau) &= 1 + \alpha \cos(2 \pi \tau) \quad (14)
\end{align*}
\]

where \(\beta\) is the exponential concentration source exponent and \(\alpha\) is the oscillatory concentration source amplitude.
3. ANALYTICAL SOLUTIONS

It is helpful to present some analytical solutions for the special cases of Eqs. (10) and (11) subject to Eq. (7) to use as standards of comparison for the numerical procedure used in the present work.

For a constant inlet concentration source \([f(\tau) = 1]\), the resulting system of partial differential equations and boundary conditions is linear. This system of equations and conditions can be solved by the Laplace transformation method assuming a semi-infinite region in order to facilitate the process of obtaining exact and approximate closed-form solutions.

A relatively simple exact solution is possible for the special case of \(N_3 = 0\) (no detachment). Equation (10) becomes like an advection-convection-reactive equation usually seen in solute transport in soil. In this case, Eq. (10) is uncoupled from Eq. (11) and its solution has the form

\[
C(\eta, \tau) = \exp \left\{ - \left[ (1 + 4N_1N_3) \right]^{1/2} - 1 \right\} \frac{\eta}{(2N_3)} \times \text{erfc} \left\{ \left[ (1 + 4N_1N_3)/(N_2N_3\tau) \right]^{1/2} (N_2\eta - \tau)/2 \right\} \\
+ \exp \left\{ \left[ (1 + 4N_1N_3)^{1/2} - 1 \right] \eta/N_3 \right\} \times \text{erfc} \left\{ \left[ (1 + 4N_1N_3)/(N_2N_3\tau) \right]^{1/2} (N_2\eta + \tau)/2 \right\} 
\]

which reduces to

\[
C(\eta, \tau) = \frac{1}{2} \left\{ \text{erfc} \left\{ \left[ (N_1 + N_2N_3)/N_5 \right]^{1/2} \times \left[ (N_1 + N_3(N_1 + N_2N_3)^2)/N_5 \right]^{1/2} \right\} + \exp \left\{ (N_1 + N_2N_3)^2\eta \right\} \times \left[ (N_1 + N_2N_3)/N_5 + \tau \right]/2 \right\} 
\]

\[
C(\eta, \tau) = \frac{1}{2} \left\{ \text{erfc} \left\{ \left[ (N_1 + N_2N_3)/N_5 \right]^{1/2} \times \left[ (N_1 + N_3(N_1 + N_2N_3)^2)/N_5 \right]^{1/2} \right\} + \exp \left\{ (N_1 + N_2N_3)^2\eta/N_1 \right\} \times \left[ (N_1 + N_2N_3)/N_5 + \tau \right]/2 \right\} 
\]

where \(\text{erfc}(\zeta) = 1 - \text{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_{\zeta}^{\infty} \exp(-\theta^2) d\theta\)

where \(\text{erf}\) and \(\text{erfc}\) are the error function and complimentary error function, respectively, and \(\theta\) is a dummy variable.

Furthermore, in the absence of diffusion \((N_3 = 0)\), Eq. (15) becomes

\[
C(\eta, \tau) = \exp(-N_1\eta)H(\tau - N_2\eta) 
\]

where \(H\) is the unit step function. It is obvious that the above equation exhibits a discontinuity at \(\eta = \tau/N_2\). For a small amount of diffusion \((N_3 \ll 1)\), this discontinuity is replaced by a narrow continuous transition layer.

For the above cases, the corresponding solution for \(S\) can be found as follows:

\[
s(\tau) = N_1 \int_{0}^{\tau} C(\eta, \tau^*) d\tau^* 
\]

An exact solution exists for the complete form of Eqs. (10) and (11) subject to Eq. (7) (for a semi-infinite region), but it involves integrals that require numerical evaluation. [See Adin and Rajagopalan (1989) for the special case of \(N_3 = 0\).] A simpler approximate solution valid for moderate and large times is obtained by Chamkha et al. (1995) based on the method of Rasmussen (1975). This solution can be written as

\[
C(\eta, \tau) = \frac{1}{2} \left\{ \text{erfc} \left\{ \left[ (N_1 + N_2N_3)/N_5 \right]^{1/2} \times \left[ (N_1 + N_3(N_1 + N_2N_3)^2)/N_5 \right]^{1/2} \right\} + \exp \left\{ (N_1 + N_2N_3)^2\eta \right\} \times \left[ (N_1 + N_2N_3)/N_5 + \tau \right]/2 \right\} 
\]

4. NUMERICAL METHOD

In their most general form, Eqs. (10) and (11) subject to Eq. (7) are nonlinear. Therefore, an analytical solution to these equations is either complex or unlikely, depending on the boundary conditions and therefore, a numerical solution is required. Many existing computer codes employ a finite-difference approach for the solution of filtration.
equations. It is logical to investigate the applicability of this methodology to the governing equations and conditions. In the present work, an implicit iterative tridiagonal finite-difference method similar to that discussed by Blottnner (1970) is employed. A two-point backward difference quotient is used to represent the dimensionless time derivative, and three-point central difference quotients are used to represent the dimensionless space derivative. The computation starts at $\tau = 0$ and marches forward in time. At each time, a system of nonlinear algebraic equations must be solved to determine the $\eta$ distributions of $C$ and $S$. An iterative procedure is employed for this purpose. At each iteration, an equivalent linear system of algebraic equations (the linearization being effected by representing some quantities by their values from the previous iteration) must be solved. These equations have a tridiagonal form and can be solved by the Potter’s method variables, which can be determined by a forward sweep in the $\eta$ direction. Then the physical variables can be found from a corresponding backward sweep. This process avoids the need for matrix inversion. Iteration is continued until convergence is obtained at a given time. The procedure moves forward for the next time. In the present work, a finite computational domain is required. For this, one can either choose a computational length such that $\eta_c \gg 1$ to approximate a semi-infinite region and use the computed solution only in the region $0 \leq \eta \leq 1$ or directly impose a nonreflecting boundary condition at the filter exit ($\eta = 1$). In all the numerical results to be presented, the computational domain was divided up into 196 points in the $\eta$ direction and 101 points in the $\tau$ direction. Variable step sizes in the $\eta$ direction and constant step sizes of 0.01 in the $\tau$ direction were employed. The initial step size and growth factor used in the $\eta$ direction were 0.001 and 1.035, respectively. These values gave $\eta_c \approx 23$ and were arrived at after performing various numerical experiments to access grid independence. Only the computed solution in the region $0 \leq \eta \leq 1$ was used. The convergence criterion required that the difference between the current and previous iterations be $10^{-7}$. The numerical method was validated by the favorable comparisons with the approximate and exact results [Eqs. (15) and (19)] obtained previously. These comparisons are shown in the graphical results presented in Figs. 1 and 2. It is seen that the numerical solutions and the approximate and exact solutions are in excellent agreement. These favorable comparisons lend confidence in the accuracy of the numerical method.

5. RESULTS AND DISCUSSION

Various numerical results are obtained and a representative set of results is presented in Figs. 2–9. These results are chosen to illustrate the influence of the time-dependent inlet concentration source functions [Eqs. (13) and (14)] on the temporal development of the solute concentration and specific deposit profiles [$C(\eta, \tau) \text{ and } S(\eta, \tau)$] as well as the time development of the filter efficiency and the specific deposit at the filter exit [$E(\tau) \text{ and } S(1, \tau)$]. Figures 2–5 are based on the inlet concentration source function given by Eq. (13), while Figs. 6–9 are based on the form given by Eq. (14).

![FIG. 1: Comparison of concentration profiles with analytical solutions [Eq. (15)]](image-url)
Figures 2–5 present the temporal development of the filter solute concentration and specific deposit profiles as well as the filter efficiency and exit specific deposit for various values of the inlet concentration source exponent $\beta$, respectively. The approximate analytical solution for $C(\eta, \tau)$ given by Eq. (15) is also presented in Fig. 2. It is clear from Fig. 2 that the approximate solution obtained from Eq. (15) is in excellent agreement with the constant diffusion solution for $\tau \geq 0.5$. All of the results shown in Figs. 2–4 capture the transition from small values of the dimensionless time $\tau$ to moderate values of $\tau (\tau = 1.0)$. In general, it is predicted that increases in the values of $\beta$ decrease the solute concentration and specific deposit for all shown time values, resulting in decreased concentration boundary-layer thicknesses. This behavior in the concentration and the specific deposit profiles and decrease in the concentration boundary-layer thickness as $\beta$ increases appear to be more significant at larger time values. Phys-
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FIG. 4: Effects of inlet concentration exponent on temporal evolution of efficiency

FIG. 5: Effects of inlet concentration exponent on exit specific deposit

ically, the decrease in the concentration boundary-layer thickness as \( \beta \) increases means that solute concentration at the exit of the filter for each time reduces. This has a direct increasing effect on the filter efficiency, as expected from its definition in Eq. (12) and consequently, the filtration time, as is shown in Fig. 4. Also, it is seen from Fig. 4 that the efficiency of the filter deteriorates as filtration time progresses. In addition, further inspection of Figs. 3 and 5 shows that for a specific value of \( \beta \), the specific deposit at the filter entrance increases as time progresses and that the same trend is predicted for the specific deposit at the filter exit. This is expected since an increase in the specific deposit of the filter implies a decrease in the filter porosity. In this way, it is understood that as the filtration time increases, the porosity of the filter decreases due to the capture of particles. Figure 5 shows that as \( \beta \) increases, the specific deposit at the filter exit decreases.
Figures 6–9 illustrate the effects of the inlet oscillatory concentration source amplitude $\alpha$ on the temporal development of the solute concentration and specific deposit profiles and the filter efficiency and exit specific deposit, respectively. The oscillatory behavior in the concentration profiles is evident in Fig. 6 especially for $\tau \geq 0.5$. For $\alpha > 0$, the solute concentration increases at the filter inlet and then decreases in the middle of the filter (for example, in the range $0.25 \leq \eta \leq 0.75$ for $\tau = 1$) and then decreases again close to the filter exit. On the other hand, the exact opposite effect is predicted for $\alpha < 0$, forming a mirror image or envelope to the corresponding cases for $\alpha > 0$ for the same frequency of oscillation. As for the specific deposit profiles, the same general behavior as that of the concentration profiles is predicted as the amplitude of oscillation $\alpha$ increases. It is clearly seen from Figs. 6–9 that for $\alpha > 0$, as $\alpha$ increases, the solute concentration level and the specific deposit decrease in the filter zone.
close to the entrance and at the filter exit ($\eta = 1$). Since
the increase in the solute concentration at the filter exit
is less than that at the inlet, the filter efficiency decreases
as $\alpha$ increases. For negative values of $\alpha$, the opposite
behavior is predicted whereby the filter efficiency increases.
As mentioned before, the exit specific deposit increases
(that is the filter porosity decreases) as $\alpha$ increases. Also,
the changes in all of $C(\eta, \tau)$, $S(\eta, \tau)$, $E(\tau)$, and $S(1, \tau)$
are more pronounced for larger values of $\tau$ than those
for small values of $\tau$. These behaviors are obvious from
Figs. 6–9.

6. CONCLUSION

A one-dimensional continuum filtration model with a
constant diffusion coefficient and superficial velocity- and
time-dependent filter inlet concentration was considered.
Two time-dependent inlet concentration sources were
considered. The first was represented by an exponentially varying function and the second by an oscillatory cosine function. The general governing equations were nonlinear and cannot be solved in closed form. However, for some special cases linear equations were obtained and exact and approximate solutions were reported. The general equations were solved numerically by an efficient, implicit, iterative tridiagonal finite-difference method. The accuracy of the numerical method was validated by favorable comparisons with the analytical solutions. Several numerical solutions based on the general model were reported. A parametric study was conducted and the results were presented graphically to illustrate interesting features of the solutions. In general, the variation of the filter inlet concentration with time produced significant changes in the solute concentration, specific deposit, and filter efficiency at most time stages compared with the constant inlet concentration source case. For an exponentially decaying inlet concentration source, it was predicted that the increased source exponent decreased the solute concentration and specific deposit for all shown time values, resulting in increases in the filter efficiency. Furthermore, for an oscillatory inlet concentration source, the filter efficiency decreased as the inlet concentration coefficient was increased. However, for negative values of the inlet concentration coefficient, the opposite behavior was predicted whereby the filter efficiency increased.

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