MHD heat and mass transfer flow of a nanofluid over an inclined vertical porous plate with radiation and heat generation/absorption

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Abstract

MHD boundary layer flow, heat and mass transfer analysis of nanofluid over an inclined vertical plate saturated by porous medium with thermal radiation, magnetic field and heat generation/absorption is investigated in the present article. By using similarity variables the governing non-linear partial differential equations are transformed into ordinary differential equations and these equations together with associated boundary conditions are solved numerically by using versatile, extensively validated, variational Finite element method. The sway of key parameters, such as, Magnetic parameter ($M$), buoyancy ratio parameter ($Nr$), Radiation parameter ($An$), Heat source/sink parameter ($Q$), Brownian motion parameter ($Nb$), thermophoretic parameter ($Nt$) and Lewis number ($Le$) on hydrodynamic, thermal and concentration boundary layers are examined in detail and the results are shown graphically. Furthermore, the impact of these parameters on local skin friction coefficient ($C_f$), rate of heat transfer ($Nu$) and rate of mass transfer ($Sh$) is also investigated. The results are compared with the works published previously and found to be excellent agreement.

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1. Introduction

A nanofluid is a fluid containing small volumetric quantities of nanometer-sized (diameter of about 1–100 nm) particles called nanoparticles. The nanoparticles used in nanofluids are naturally made of metals (Cu, Al), (or) oxides (TiO$_2$, Al$_2$O$_3$, CuO, SiO$_2$), carbides (SiC), nitrides (SiN, AlN), (or) nonmetals (graphite, carbon nanotubes). The convectional heat transfer fluids like water, oil and ethylene glycol mixture have low heat transfer abilities due to their less thermal conductivity. In general, the thermal conductivity of the metals is three times more than that of general fluids, so it is admissible to mix the two substances to produce a heat transfer medium that behaves like a fluid but has the metal's thermal conductivity. To improve the thermal conductivity of these fluids nano/micro-sized particle materials are suspended in liquids. Several theoretical and experimental investigations have been made to enhance the thermal conductivity of these fluids; Choi [1] was the first among all who introduced a new type of fluid called nanofluid while doing research on new coolants and cooling technologies. Eastman et al. [2] have noticed in an experiment that the thermal conductivity of the base fluid (water) has increased up to 60% when CuO nanoparticles are added to the base fluid with volume fraction 5%. This enhancement is because of increasing surface area of the base fluid due to the suspension of nanoparticles. Eastman et al. [3] have revealed that the thermal conductivity has increased 40% when copper nanoparticles with the volume fraction less than 1% are added to the ethylene glycol or oil. Choi et al. [4] have reported that there is 150% enhancement in the thermal conductivity when carbon nanotubes are added to the ethylene glycol or oil. In addition, Xie et al. [5] have observed that Al$_2$O$_3$-ethylene glycol based nanofluid thermal conductivity is increased in the range 25–30% when Alumina nanoparticles are added. Buongiorno [6] has reported in his bench mark study using scale analysis that, there are seven possible mechanisms which increases the thermal conductivity of the nanofluid, such as, nanoparticle size, inertia, particle agglomeration, Magnus effect, volume fraction of the nanoparticle, Brownian motion and thermophoresis. Among all these they found that Brownian motion and thermophoresis are significant slip mechanisms in nanofluids.
We can see many research papers in literature which deals the heat and mass transfer characteristics of nanofluids by considering Brownian motion and thermophoresis effects into the account. Nield and Kuznetsov [7] have discussed the Cheng-Mincowycz problem for natural convection boundary-layer flow in a porous medium saturated nanofluid. Kuznetsov and Nield [8] deliberated the impact of Brownian motion and thermophoresis on natural convection boundary layer flow, heat and mass transfer of a nanofluid past a vertical plate. Khan and Pop [9] have discussed boundary layer flow of a nanofluid over a stretching sheet. Chamkha et al. [18] have deliberated mixed convection flow over a stretching sheet with Soret and Dufour effects and non-uniform heat generation/absorption. Aly [17] has analyzed thermal radiation and magnetic field effect on free convective boundary layer flow over a stretching sheet filled with nanofluid by taking magnetic field effect into the account.

Thermal radiation plays very significant role in the surface heat transfer when convection heat transfer is very small. It has applications in manufacturing industries, the design of reliable equipment’s, nuclear plants, gas turbines, aircraft, missiles, satellites, space vehicles, space technology and procedure relating high temperature. Gireesh et al. [14] have studied the impact of radiation and heat source/sink on mixed convective Dusty fluid over a stretching sheet. Rashidi et al. [16] have reported natural convection boundary layer analysis of micropolar fluid through porous medium with radiation and chemical reaction. Rashidi et al. [16] have presented the effect of thermal radiation on mixed convective flow and heat transfer of Visco-elastic fluid over porous wedge. Aly [17] has analyzed thermal radiation and magnetic field effect on boundary layer flow of nanofluid over a stretching sheet through porous medium with suction/injection and heat source/sink. Chamkha et al. [18] have deliberated mixed convection flow over a permeable surface embedded in a porous medium with thermal radiation. Chamkha et al. [19] has studied radiation effect on mixed convective flow through porous medium over a wedge filled with nanofluid. Noghrehabadi et al. [20] have presented natural convection flow, heat and mass transfer of nanofluid over a vertical semi-infinite vertical flat plate. Pournima et al. [13] have presented the effect of radiation on free convective boundary layer flow over a stretching sheet filled with nanofluid by taking magnetic field effect into the account.

Magnetofluid nanofluids have specific applications in biomedicine, optical modulators, magnetic cell separation, magneto-optical wavelength filters, solar float separation, nonlinear optical materials, hyperthermia, optical switches, drug delivery, optical gratings etc. A magnetic nanofluid has both the liquid and magnetic properties. The used magnetic field influences the suspended particles and reorganizes their concentration in the fluid regime which powerfully influences the heat transfer analysis of the flow. Magnetofluid nanofluids are useful to guide the particles up the blood stream to a tumor with magnets. This is due to the fact that the magnetic nanoparticles are regarded more adhesive to tumor cells than non-malignant cells. Such particles absorb more power than microparticles in alternating current magnetic fields tolerable in humans i.e. for cancer therapy. Sudarsana Reddy et al. [11] have analyzed finite element analysis of unsteady MHD heat and mass transfer flow over stretching sheet with Soret and Dufour effects and non-uniform heat generation/absorption. Hamad et al. [12] have discussed magnetic field effect on free convection flow of a nanofluid over a vertical semi-infinite vertical flat plate. Poornima et al. [13] have presented the effect of radiation on free convective boundary layer flow over a stretching sheet filled with nanofluid by taking magnetic field effect into the account.

Theoretical models have been developed to predict the force on a particle in the magnetic field. Some of these models have been discussed in detail by Hamad et al. [12]. The force on a particle due to an electric field is given by the equation:

$$\text{F} = q \text{E}$$

where $q$ is the electric charge and $\text{E}$ is the electric field. The force on a particle due to a magnetic field is given by the equation:

$$\text{F} = \mu_0 \text{v} \times \text{B}$$

where $\mu_0$ is the permeability of free space, $\text{v}$ is the velocity of the particle, and $\text{B}$ is the magnetic field.

The force on a particle due to a thermal gradient is given by the equation:

$$\text{F} = -k \nabla T$$

where $k$ is the thermal conductivity and $\nabla T$ is the temperature gradient.

These forces are superimposed to determine the total force on the particle. The particle will move in the direction of the resultant force.
Sudarsana Reddy et al. [24,25] have presented MHD natural convection boundary layer flow, heat and mass transfer characteristics of two different nanofluids over a rotating disk, vertical cone, respectively, with chemical reaction. The problem of boundary layer, heat and mass transfer flows through porous medium over an inclined plate has received much interest in recent years because of its engineering and industrial applications. Chamkha et al. [26] have reported the natural convection from an inclined plate through a variable porosity porous medium with magnetic field and solar radiation. They suggested that increasing the plate inclination angle resist the motion of the fluid which causes an enhancement in the temperature of the fluid. Alam et al. [27] has deliberated the flow, heat and mass transfer analysis over an inclined plate by taking heat generation and thermophoresis. Very recently, Puneet Rana et al. [28] have presented finite element and finite difference numerical solution for mixed convection flow of a nanofluid along a semi-infinite inclined flat plate through porous medium in the presence of Brownian motion and thermophoresis. They reported that temperature and concentration profiles are remarkably influenced with plate inclination parameter. The different aspects and applications of nanofluid flow over various geometries have been investigated by several authors [29–36].

To the ability of authors’ knowledge, no studies have been presented in the literature to discuss MHD mixed convection heat and mass transfer characteristics of nanofluid over an inclined vertical flat plate by taking thermal radiation and heat generation/absorption into the account. Hence, we made an attempt to discuss the problem here. The transformed conservation equations with the corresponding boundary conditions are solved numerically by using variational, extensively validated, variational finite element method.

2. Mathematical analysis

We consider steady, laminar, two dimensional, incompressible, mixed convection flow, heat and mass transfer of a nanofluid over an inclined plate embedded through porous medium, with an acute angle $\alpha$ to the vertical, illustrated in Fig. 1. The two slip effects of Brownian motion and thermophoresis is considered in this problem. The coordinate system is such that x-axis is along the plate in the direction of motion and y-axis is perpendicular to it. The plate is maintained at constant temperature $T_w$ and concentration $C_w$. The temperature and concentration of the ambient fluid are assumed to be $T_\infty$ and $C_\infty$, respectively. A constant magnetic field of intensity $B_0$ is applied in the y-axis direction.

![Fig. 1. Physical model and coordinate system.](image)

Imposing magnetic field into the electrically conducting fluid creates a resistive force in the negative x-direction and is called Lorentz force. Based on the reference works of [8,28], by using the Oberbeck-Boussinesq approximation, the governing equations describing the steady-state conservation of mass, momentum, energy as well as conservation of nanoparticles for nanofluids in the presence of thermal radiation and other important parameters take the following form:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}
\]

\[
\frac{\partial p}{\partial y} = 0 \tag{2}
\]

\[
\mu_f \frac{1}{k} u = -\frac{\partial p}{\partial x} + f \left[ (1 - C_w) \rho_f n (T - T_w) - (\rho_p - \rho_{nf}) (C - C_w) \right] \times \cos \alpha - \frac{\sigma_r B_0^2 u}{\rho_f} \tag{3}
\]

\[
\left( \frac{\partial T}{\partial x} - \nu \frac{\partial T}{\partial y} \right) - \frac{k_m}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial y^2} + \frac{\partial (\rho c_p)_{nf}}{(\rho c_p)_{nf}} \left[ D_b \frac{\partial C}{\partial y} + \frac{D_t}{T_w} \frac{\partial T}{\partial y} \right]^2 - \frac{1}{(\rho c_p)_{nf}} \frac{\partial}{\partial y} \left( q_r \right) \tag{4}
\]

\[
1 \left( \frac{\partial T}{\partial x} - \nu \frac{\partial T}{\partial y} \right) = \frac{D_t}{T_w} \frac{\partial^2 T}{\partial y^2} \tag{5}
\]

The boundary conditions based on the problem description are as follows:

\[
u = 0, \quad T = T_w, \quad C = C_w \text{ at } y = 0 \tag{6}
\]

\[
u, \quad u \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty \text{ at } y \rightarrow \infty \tag{7}
\]

$p$ may be eliminated from Eqs. (2) and (3) by cross-differentiation and the continuity Eq. (1) will be satisfied introducing a stream function $(\psi)$ defined by the Cauchy-Riemann equations:

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{8}
\]

By substituting, Eq. (8) in Eqs. (3)–(5), we get

\[
\frac{\partial^2 \psi}{\partial y^2} = \left[ \left( 1 - C_w \right) \rho_f n g k \frac{\partial T}{\partial y} + \frac{(\rho_p - \rho_{nf}) g k \partial C}{\mu} \cos \alpha \right] - \frac{\sigma_r B_0^2 \partial \psi}{\rho_f} \tag{9}
\]

\[
\frac{\partial \psi}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial y} = \frac{k_m}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial y^2} + \frac{\partial (\rho c_p)_{nf}}{(\rho c_p)_{nf}} \left[ D_b \frac{\partial C}{\partial y} + \frac{D_t}{T_w} \frac{\partial T}{\partial y} \right]^2 \tag{10}
\]

\[
\frac{1}{\varepsilon} \frac{\partial (\rho c_p)_{nf} \partial \psi}{\partial y} + \frac{\partial (\rho c_p)_{nf} \partial \psi}{\partial x} = D_t \frac{\partial^2 C}{\partial y^2} + \frac{D_t}{T_w} \frac{\partial T}{\partial y} \tag{11}
\]

The following similarity transformations are introduced to simplify the mathematical analysis of the problem:

\[
\eta = \frac{y}{x \sqrt{Pe_x}}, \quad \phi = \frac{\eta}{x \sqrt{Pe_x}}, \quad \theta(\eta) = \frac{T - T_w}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \tag{12}
\]

where $Pe_x = \frac{k_m x}{\mu}$. 
By using Rosseland approximation for radiation, the radiative heat flux \( q_r \) is defined as

\[
q_r = -\frac{4G_c^* \partial T^4}{3K^* \partial Y}.
\]  

(13)

We assume that the temperature differences within the flow are such that the term \( T^4 \) may be expressed as a linear function of temperature. This is accomplished by expanding \( T^4 \) in a Taylor series about a free stream temperature \( T_\infty \) as follows:

\[
T^4 = T^4_\infty + 4T^3_\infty (T - T_\infty) + 6T^2_\infty (T - T_\infty)^2 + \cdots
\]  

(14)

Neglecting higher-order terms in the above Eq. (14) beyond the first degree in \( (T - T_\infty) \), we get

\[
T^4 \approx 4T^3_\infty T - 3T^4_\infty.
\]  

(15)

Thus substituting Eq. (15) in Eq. (13), we get

\[
q_r = -\frac{16T^3_\infty \sigma^* \partial T}{3K^* \partial Y}.
\]  

(16)

Using the similarity variables (12), Eqs. (9) to (11) together with boundary conditions (6) and (7) reduce to

\[
f'' = \frac{R_s}{Pe} \left( \vartheta' - Nt \varphi' \right) \cos(\chi) + Mf''
\]  

(17)

\[
(1 + \frac{4}{3}An) \vartheta'' + \frac{1}{2} f'' + Nt \varphi' \varphi'' + Nt(\vartheta')^2 - Q \vartheta = 0
\]  

(18)

\[
\varphi'' + \frac{1}{2} Le f \varphi' + \frac{Nt}{Nt} \vartheta = 0
\]  

(19)

The transformed boundary conditions are

\[
\eta = 0, \quad f = 0, \quad \vartheta = 1, \quad \varphi = 1.
\]

\[
\eta = \infty, \quad f = 1, \quad \vartheta = 0, \quad \varphi = 0.
\]  

(20)

where prime denotes differentiation with respect to \( \eta \), and the key thermophysical parameters dictating the flow dynamics are defined by

\[
Nr = \frac{(\rho_0 - \rho_\infty)(C_w - C_\infty)}{\rho_\infty (T_w - T_\infty)(1 - C_\infty)}, \quad Nb = \frac{\rho(\rho C_p)D_\Theta(C_w - C_\infty)}{(\rho C_p)z_m},
\]

\[
Nt = \frac{\rho(\rho C_p)D_\Theta(T_w - T_\infty)}{(\rho C_p)z_m}, \quad Le = \frac{x_m}{\Delta D},
\]

\[
Ra_s = \frac{(1 - C_\infty)kg \rho_\infty (T_w - T_\infty)}{\mu z_m}, \quad Pe_x = \frac{U_x x}{z_m}, \quad Ra = \frac{Ra_s}{Pe_x},
\]

\[
Q = \frac{x^2}{Pe_x z_m}, \quad An = \frac{4T^3_\infty \sigma^*}{K^* z_m}, \quad M = \frac{\sigma^* x^2}{\rho Pe_x^{1/2}}.
\]

Quantities of practical interest in this problem are local Nusselt number (\( Nu_l \)), local Sherwood number (\( Sh_l \)) are defined as

\[
Nu_l = \frac{xq_m}{k(T_w - T_\infty)}, \quad Sh_l = \frac{xq_m}{D_\Theta(C_w - C_\infty)}
\]  

(21)

By using the above equations we get the following reduced Nusselt number (\( Nu_r \)) and reduced Sherwood number (\( Sh_r \)) and are modeled as

\[
(Pe_x)^{-1/2} Nu_l = -\vartheta(0), \quad (Pe_x)^{-1/2} Sh_l = -\varphi(0),
\]  

(22)

Eqs. (17)-(19) are extremely non-linear, so finite-element method [37-40] has been implemented to solve these non-linear equations. The very important aspect in this numerical procedure is to select an approximate finite value of \( \eta_{\infty} \). So, in order to estimate the relevant value of \( \eta_{\infty} \), the solution process has been started with an initial value of \( \eta_{\infty} = 4 \), and then Eqs. (17)-(19) are solved together with boundary conditions (20). We have updated the value of \( \eta_{\infty} \) and the solution process is continued until the results are not affected with further values of \( \eta_{\infty} \). The choice of \( \eta_{\infty} = 8 \) and \( \eta_{\infty} = 5 \) for temperature and concentration have confirmed that all the numerical solutions approach to the asymptotic values at the free stream conditions.

3. Numerical method of solution

3.1. The finite-element method

The finite-element method (FEM) is such a dominant method for solving ordinary differential equations and partial differential equations. The basic idea of this method is dividing the whole domain into smaller elements of finite dimensions called finite elements. This method is such a good numerical method in modern engineering analysis. The steps involved in this method are described below.

(i) Finite-element discretization

In the finite element discretization the entire interval is divided into a finite number of subintervals and this subinterval is called an element. The set of all these elements is called the finite-element mesh.

(ii) Generation of the element equations

a. Variational formulation of the mathematical model over the typical element (an element from the mesh) is performed.

b. We assume an approximate solution of the above variational problem, and with the help of this solution we can generate element equations.

c. Using interpolating polynomials the stiffness matrix is constructed.

(iii) Assembly of element equations

All the algebraic equations are assembled with the help of inter element continuity conditions. This result a large number of algebraic equations called global finite-element model and it represents the whole domain.

(iv) Imposition of boundary conditions

The boundary conditions which represent the flow model are imposed on the assembled equations.

(v) Solution of assembled equations

By using any one of the numerical technique we can solve all the assembled equations.

3.2. Variational formulation

The variational form related with Eqs. (17)-(19) over a typical linear element (\( \eta_r, \eta_{r+1} \)) is given by

\[
\int_{\eta_r}^{\eta_{r+1}} w_1(f'' - Ra(\vartheta' + Nt \varphi') \cos(\chi) - Mf'')d\eta = 0
\]  

(23)

\[
\int_{\eta_r}^{\eta_{r+1}} w_2 \left( (1 + \frac{4}{3}An) \vartheta'' + \frac{1}{2} f'' + Nt \varphi' \varphi'' + Nt(\vartheta')^2 - Q \vartheta \right) d\eta = 0
\]  

(24)

\[
\int_{\eta_r}^{\eta_{r+1}} w_3 \left( \varphi'' + \frac{1}{2} Le f \varphi' + \frac{Nt}{Nt} \vartheta \right) d\eta = 0
\]  

(25)
where \( w_1, w_2, \) and \( w_3 \) are weighted functions and may be regarded as the variations in \( f, \theta, \) and \( \phi, \) respectively.

### 3.3. Finite-element formulation

The finite-element form may be attained from above equations by replacing finite-element approximations of the form

\[
\psi = \sum_{i=1}^{3} \phi_i \psi_i, \quad \theta = \sum_{j=1}^{3} \theta_j \psi_j, \quad \phi = \sum_{j=1}^{3} \phi_j \psi_j
\]

with \( w_1 = w_2 = w_3 = \psi_i, \quad (i = 1, 2, 3). \)

where \( \psi_i \) are the shape functions for a typical element \( (\eta_e, \eta_{e+1}) \) and are defined as

\[
\psi_1 = \frac{(\eta_{e+1} - \eta_e - 2\eta)(\eta_{e+1} - \eta)}{(\eta_{e+1} - \eta_e)^2}, \quad \psi_2 = \frac{4(\eta - \eta_e)(\eta_{e+1} - \eta)}{(\eta_{e+1} - \eta_e)^2}, \quad \psi_3 = \frac{(\eta_{e+1} - \eta_e - 2\eta)(\eta - \eta_e)}{(\eta_{e+1} - \eta_e)^2}, \quad \eta_e < \eta < \eta_{e+1}.
\]

The finite element model of the equations thus given is formed by

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
f \\
\theta \\
\phi
\end{bmatrix}
= \begin{bmatrix}
\{r_1\} \\
\{r_2\} \\
\{r_3\}
\end{bmatrix}
\]

where \( [K^{ee}] \) and \( [p^{ee}] \) are defined as

\[
K_{11} = - \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \, d\eta + M \int_{\eta_e}^{\eta_{e+1}} \psi_i \, d\eta,
\]

\[
K_{12} = - Ra \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta} \, d\eta,
\]

\[
K_{13} = Ra + Nr \int_{\eta_e}^{\eta_{e+1}} \psi_i \, d\eta,
\]

\[
K_{21} = \frac{1}{2} \int_{\eta_e}^{\eta_{e+1}} \psi_i \, d\eta,
\]

\[
K_{22} = \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \, d\eta + Nt \int_{\eta_e}^{\eta_{e+1}} \psi_i \frac{\partial \psi_i}{\partial \eta} \, d\eta + Q \int_{\eta_e}^{\eta_{e+1}} \psi_i \, d\eta,
\]

\[
K_{23} = Nb \int_{\eta_e}^{\eta_{e+1}} \psi_i \, d\eta, \quad K_{31} = \frac{1}{2} Le Nb \int_{\eta_e}^{\eta_{e+1}} \psi_i \, d\eta,
\]

\[
K_{32} = - Nt \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \, d\eta,
\]

\[
K_{33} = Nb \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \, d\eta.
\]

\[
r_i = 0, \quad r_1 = \left( \psi \frac{\partial \psi_i}{\partial \eta} \right)_{\eta=\eta_e}, \quad r_3 = \left( \psi \frac{\partial \psi_i}{\partial \eta} + \frac{\partial \psi_i}{\partial \eta} \right)_{\eta=\eta_e}.
\]

where \( \tilde{f} = \sum_{j=1}^{3} \int_{\eta_e}^{\eta_{e+1}} \tilde{f} \, d\eta, \quad \tilde{\theta} = \sum_{j=1}^{3} \int_{\eta_e}^{\eta_{e+1}} \tilde{\theta} \, d\eta, \quad \tilde{\phi} = \sum_{j=1}^{3} \int_{\eta_e}^{\eta_{e+1}} \tilde{\phi} \, d\eta.

After assembly of element equations, we get the system of strongly non-linear equations and are solved using a robust iterative scheme. The system is linearized by incorporating the functions \( f, \theta, \) and \( \phi, \) which are assumed to be known. After imposing the boundary conditions, we get the less number of non-linear equations and are solved using Gauss elimination method. The computer program of the algorithm was executed in MATHEMATICA 10.0 running on a PC. To investigate the sensitivity of the solutions to mesh density, we have performed the grid invariance test (Cu – water nanofluid) for temperature and concentration distributions and are shown in Table 1. It is observed from this table that in the same domain the accuracy is not affected, even the number of elements increased, by decreasing the size of the elements.

#### 4. Results and discussion

Comprehensive numerical computations were directed for different values of the key parameters that describe the flow characteristics, and the results are showed graphically and in tabular form. Selected graphical profiles are presented in Figs. 2–18. The results are compared for Nusselt number \((-\theta'(0))\) and Sherwood number \(-\phi'(0))\) with several values of \((Le, \, (Nt, \) and \((Nb)\) with the works reported by Puneet Rana et al. [28] and are presented in Table 2. Thus, it is seen from Table 1 that the numerical results are in close agreement with those published previously.

The influence of magnetic field parameter \((M)\) on thermal and solutal boundary layers is depicted in Figs. 2 and 3. It is noticed from these figures that the thermal boundary layer thickness as well as the solutal boundary layer thickness enhances with enhance in the values of \((M)\). This is because of the fact that, the presence of magnetic field in an electrically conducting fluid produces a force called Lorentz force, this force acts against the flow direction causes the depreciation in velocity profiles, and at the same time, to overcome the drag force imposed by the Lorentzian retardation the fluid has to perform extra work; this supplementary work can be converted into thermal energy which increases the temperature of the fluid (Fig. 2) and also increases the concentration profiles (Fig. 3).

Figs. 4 and 5 depict the effect of mixed convection parameter \(Ra\) on temperature and concentration distributions. It is analyzed that both temperature and concentration profiles decelerates with the increasing values of mixed convection parameter \((Ra)\). This is because of the fact that buoyancy ratio parameter \((Nr)\) influence is less in the boundary layer regime than the mixed convection parameter, so that, there is retardation in the thickness of thermal and solutal boundary layers. Furthermore, the temperature and concentration profiles increase when \(Ra = 0\) (forced convection) because of no buoyancy forces, and both profiles retards with the increasing values of \(Ra\).

Figs. 6 and 7 illustrate the effect of buoyancy ratio parameter \((Nr)\) on temperature and concentration distributions. The temperature profiles of the fluid increases with the increasing values of buoyancy ratio parameter \((Nr)\). This is from the reality that higher the values buoyancy ratio parameter enhances the fluid temperature, so that thermal boundary layer thickness is increased (Fig. 6). The concentration profiles also enhances throughout the fluid region for different values of the buoyancy ratio parameter \((Nr)\). This is because of the fact that solutal boundary layer thickness elevates with increasing values of \((Nr)\) (Fig. 7).

The variation in temperature and concentration profiles for different values of plate inclination angle \((\chi)\) is depicted in Figs. 8 and 9. It is noticed from Fig. 8 that, an increase in the plate inclination angle \((\chi)\) resist the motion of the fluid which causes an enhancement in the temperature of the fluid (Fig. 8). It is also observed that the concentration distributions are increased with increase in the values of \((\chi)\) (Fig. 9).

This elevation in temperature and concentration profiles is because of the reduction in buoyancy ratio term \(g(\beta(T - T_{\infty}) - \beta(C - C_{\infty}))\) \(\cos(\pi)\) in the momentum equation with increasing values of \(\chi\). It is also noticed that the maximum buoyancy force occur for \(\chi = 0\) (vertical plate) and there is no buoyancy term when \(\chi = \pi/2\) (horizontal plate). From these two figures we conclude that both thermal boundary layer and solutal boundary layers elevates with the plate inclination angle \(\chi\).

Temperature and concentration distributions for different values of heat source/sink parameter \((Q)\) are illustrated in Figs. 10 and 11. It is observed that temperature in thermal boundary layer increases with the increase in heat generation parameter \((Q > 0)\), whereas thermal boundary layer thickness decelerates with the heat absorption parameter \((Q < 0)\) as shown in Fig. 10. This is
due to the fact that, increasing the values of \( (Q > 0) \) in the boundary layer region creates energy as a result temperature of the fluid enhances, whereas decreasing the values of \( (Q < 0) \) absorbs the temperature of the fluid and is causes the deceleration in the temperature of the fluid. However, the similar opposite trend is noticed in the concentration boundary layer.

The effect of thermal radiation parameter \((An)\) on temperature and concentration profiles is illustrated in Figs. 12 and 13. It is noticed from Fig. 12 that, the thermal boundary layer thickness is enhanced with the higher values of \((An)\) in the entire flow region. This is due to the fact that imposing thermal radiation into the flow warmer the fluid, which causes an increment in the temperature of the fluid. However, there is a deceleration in the concentration boundary layer thickness with the increasing values of \((An)\).

The effect of Brownian motion parameter \((Nb)\) on temperature and concentration profiles is illustrated in Figs. 14 and 15. It is noticed that, with the increasing values of Brownian motion parameter \((Nb)\) temperature profiles enhanced as shown in Fig. 14, whereas, fluid concentration profiles depreciates with enhancing values of \((Nb)\) (Fig. 15). This type of tendency in temperature and concentration is same as in the case of general heat
transfer fluids. Many theoretical studies suggested that, enhancement in thermal conduction of the fluid by Brownian motion of nanoparticles is because of any one of the following reasons; either nanoparticles direct effect on the fluid flow that transport heat or an indirect influence of adjacent different nanoparticles due to micro-convection of the fluid.

Variations in the thickness of thermal and solutal boundary layers for diverse values of thermophoretic parameter ($N_t$) is depicted in Figs. 16 and 17. It is noticed from these figures that both temperature and concentration profiles elevates in the boundary layer region for the higher values of thermophoretic parameter ($N_t$). This is from the reality that particles near the hot
surface create thermophoretic force; this force enhances the temperature and concentration of the fluid in the fluid region. We noticed from Fig. 16 that the temperature differences are small, this is because of the fact that thermophoretic parameter is a nanoscale parameter so that its influence is relatively less.

The impact of Lewis number ($Le$) on solutal boundary layer profiles is plotted in Fig. 18. It is observed that the concentration distributions decelerate with higher values of Lewis number ($Le$) in the entire flow region. By the definition, Lewis number can be defined as the ratio of thermal diffusivity to mass diffusivity. Lewis number increasing means higher the thermal diffusivity and lower the mass diffusivity, as a result thinner the concentration boundary layer.
The values of Nusselt number $-\theta(0)$ and Sherwood number $-\psi(0)$ for different values of the important parameters are presented in Table 3. It is clear that the dimensionless heat transfer rates enhanced with increase in magnetic parameter ($M$), while dimensionless mass transfer rates retards with ($M$). The increase in mixed convection parameter ($Ra$) decelerates the rates of dimensionless heat transfer whereas increases the mass transfer rates. It is evident that the dimensionless heat transfer rates boosted with higher values of heat source parameter ($Q$). However, it is found that dimensionless mass transfer rates sinks with higher values of ($Q$). The dimensionless heat transfer rates deteriorate with the higher values of thermal radiation parameter ($An$), whereas reverse trend is noticed in the rates of dimensionless mass transfer in the flow region. It is clear from the table that the rates of dimensionless heat transfer declines with the increasing values both Brownian motion parameter ($Nb$) and thermophoresis parameter ($Nt$). However, the dimensionless mass transfer rates increases with the higher values of Brownian motion parameter, while it retards with increasing values of ($Nt$).

Table 3

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<th>Parameter</th>
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<td>$Nt$</td>
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5. Conclusions

In this article, a theoretical study is implemented to examine the heat and mass transfer characteristics of a MHD nanofluid along an inclined vertical plate embedded in a porous medium with thermal radiation, and heat generating source. The governing partial differential equations are transformed into the set of nonlinear ordinary differential equations using similarity variables. The resulting ordinary differential equations together with boundary conditions are solved using Finite element method. The influence of magnetic field and thermal radiation parameters on temperature and concentration profiles were analyzed in detail and are presented graphically. In addition to that the reduced Nusselt and Sherwood numbers were calculated for different values of important parameters and shown in tabular form. The highlights of the present problem can be summarized as follows.

(i) Temperature and concentration profiles rises with higher values of Magnetic field parameter ($M$). However, the gradient of temperature enhances whereas the gradient of diffusion decelerates with increasing values of ($M$).

(ii) Both parameters Brownian motion ($Nb$) and thermophoresis ($Nt$) increases the thickness of the thermal boundary layer in the flow region. Whereas local Nusselt number reduces with higher values of ($Nb$) and ($Nt$).

(iii) Increasing the plate inclination angle ($\alpha$) elevates the temperature and concentration distributions.

(iv) Temperature profiles enhances with increasing values of thermal radiation parameter ($An$), but dimensionless temperature rates denigrates and mass transfer rates enhances with ($An$).

(v) The temperature profiles elevate with heat generation parameter ($Q > 0$), whereas retards with heat absorption parameter ($Q < 0$).