Radiative nanofluid flow and heat transfer between parallel disks with penetrable and stretchable walls considering Cattaneo–Christov heat flux model

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Abstract
In this work, we explore the unsteady squeezing flow and heat transfer of nanofluid between two parallel disks in which one of the disks is penetrable and the other is stretchable/shrinkable, in the presence of thermal radiation and heat source impacts, and considering the Cattaneo–Christov heat flux model instead of the more conventional Fourier's law of heat conduction. A similarity transformation is utilized to transmute the governing momentum and energy equations into nonlinear ordinary differential equations with the proper boundary conditions. The achieved nonlinear ordinary differential equations are solved by the Duan–Rach Approach (DRA). This method modifies the standard Adomian Decomposition Method by evaluating the inverse operators at the boundary conditions directly. The impacts of diverse active parameters, such as the suction/injection parameter, the solid volume fraction, the heat source parameter, the thermal relaxation parameter,

Nomenclature:

A, Suction/injection parameter; A1, A2, A3, A4, Dimensionless constants; ADM, Adomian Decomposition Method; C, Stretching/shrinking parameter; Cp, Skin fraction coefficient; Cₚ, Specific heat (j kg⁻¹K⁻¹); DRA, Duan–Rach Approach; f, Dimensionless velocity; Hs, Heat source parameter; k, Thermal conductivity; k*, Mean absorption coefficient; N, Radiation parameter; Nu, Nusselt number; P, Pressure term; Pr, Prandtl number; q_rad, Radiative heat flux; S, Squeeze number; T, Temperature (K); u, Velocity component in r direction (m s⁻¹); w, Velocity component in z direction (m s⁻¹); γ, Thermal relaxation parameter; θ, Dimensionless temperature

Subscripts:
f, Base fluid; nf, Nanofluid; s, Nanosolid particles

Greek symbols:

α, Rate of squeezing; η, Dimensionless variable; μ, Dynamic viscosity; ν, Kinematic viscosity; ρ, Density; σ*, Stefan–Boltzmann constant; φ, Solid volume fraction
and the radiation parameter on flow and heat transfer traits are examined. In addition, the value of the Nusselt number is calculated and portrayed through figures.

**KEYWORDS**
Cattaneo–Christov heat flux model, Duan–Rach Approach (DRA), nanofluid, penetrable and stretchable walls, squeezing flow

1 | INTRODUCTION

One of the most important issues in industrial and engineering systems is the heat transfer phenomenon. We know that temperature differences between media or within a medium leads to such phenomenon. In the past two centuries, Fourier's law of heat conduction has been applied to study heat transfer traits. The primary difficulty of Fourier's law is that it presents a parabolic energy equation, which means that any initial disturbance is sensed promptly all over the whole system. To manage this difficulty, Cattaneo\(^1\) modified Fourier's law of heat conduction by factoring thermal relaxation time into it. Consequently, in Cattaneo's model, the heat transport is conducted via thermal wave propagation with a specified speed. Thereafter, Christov\(^2\) improved Cattaneo's model by introducing thermal relaxation time in terms of Oldroyd's upper-convicted derivative for the sake of achieving the material-invariant formulation. Lately, many authors have applied the Cattaneo–Christov heat flux model instead of the more conventional Fourier's law of heat conduction to their studies. Liu and colleagues\(^3\) analyzed heat conduction with a fractional Cattaneo–Christov upper-convective derivative flux model. Their outcomes indicate that the fractional parameters, time and location parameters, relaxation parameter, weight coefficient, and convection velocity all have remarkable impacts on heat transfer characteristics. Hayat and colleagues\(^4\) explored the magnetohydrodynamic (MHD) flow of an Oldroyd-B fluid over a stretching surface with homogeneous–heterogeneous reactions. They applied the Cattaneo–Christov heat flux model for the simulation of the energy equation instead of Fourier's law of heat conduction. Salahuddin and colleagues\(^5\) investigated MHD flow in the Cattaneo–Christov heat flux model for Williamson fluid over a stretching sheet with variable thicknesses. Their results illustrate that the velocity profile abates, whereas the temperature profile ascends with a rise in the Hartmann number. Shahid and colleagues\(^6\) explored Radiative Maxwell viscoelastic magnetized flow from a stretching permeable sheet with the Cattaneo–Christov heat flux model. They proved that when the suction/injection parameter rises, then the velocity of the fluid abates strikingly along the \(u\) component of velocity, whereas the converse behavior is computed for the \(v\) component of velocity. Abbasi and colleagues\(^7\) studied the Cattaneo–Christov heat flux model for a laminar boundary layer flow of an incompressible Oldroyd-B fluid over a linearly stretching sheet. They reported that the temperature and the thermal boundary layer thickness are smaller in the Cattaneo–Christov heat flux model than those in Fourier's law of heat conduction.

In recent years, there has been a rise in the use of nanofluids to raise the heat transfer rate in engineering systems with the low thermal conductivity of the commonplace fluids used in those systems. This is due to the fact that those nanofluids have a thermal conductivity higher than commonplace fluids owing to the metallic nanoparticles suspended in the fluid. In this regard, vast research has been carried out on the role of nanofluids in fortifying heat transfer rate. Rashidi and colleagues\(^8\) investigated flow and heat transfer of a nanofluid over a stretching sheet in the presence of a transverse magnetic
field with thermal radiation and buoyancy impacts. They demonstrated that the ascent of the buoyancy parameter raises the velocity and abates the nanofluid temperature. The combined impacts of thermophoresis and thermal radiation on Williamson nanofluid over a porous stretching sheet was examined by Bhatti and Rashidi. They found that the magnitude of fluid is greater for higher values of Williamson fluid parameter and porosity parameter. Rashidi and colleagues explored magnetic field impacts on the flow of a Burgers’ nanofluid over an inclined wall. Their results show that the concentration profile and concentration boundary layer are ascending functions of the thermophoretic parameter. Ellahi and colleagues reviewed flow and heat transfer of a nanofluid in the neighborhood of a stagnation point flow in the presence of mixed convection. Their outcomes indicate that the temperature of the nanofluid increases with a rise in the volume friction, chemical dimensions and the radius of gyration. Akbarzadeh and colleagues studied nanofluid flow and force convective heat transfer inside a wavy channel. They reported that the sensitivities of the average Nusselt number to the Reynolds number and channel aspect ratio rise with an increase in the channel aspect ratio. Ellahi and colleagues investigated particle shape impacts on the Marangoni convection boundary layer flow of a nanofluid. They proved that the interface velocity is reduced by augmenting the particle volume friction and volume concentration of ethylene glycol in the water. Hosseinzadeh and colleagues studied the heat and mass transfer of MHD squeezing nanofluid flow between parallel plates by analytical and numerical methods. They found that that temperature boundary layer thickness ascends by amplification of Brownian motion and Thermophoresis parameters, while it abates by ascending the other active parameters. Simultaneous effects of coagulation (blood clot) and variable magnetic fields on peristaltically induced motion of non-Newtonian Jeffrey nanofluids containing gyrotactic microorganisms through an annulus was reviewed by Bhatti and colleagues. They found that the velocity of a fluid reduces near the walls owing to an augmented height of the clot. Sheikholeslami and colleagues explored nanofluid flow and heat transfer traits between two horizontal plates in a rotating system. They realized that for both suction and injection processes, the heat transfer rate at the surface ascends with an increase in the nanoparticle volume fraction, Reynolds number, and injection/suction parameter, while it declines with the power of rotation parameter. Sheikholeslami and Ganji studied three dimensional two phase simulation of nanofluid flow and heat transfer between two horizontal parallel plates. They proved that the concentration boundary layer thickness raises with the rise in the Thermophoretic parameter and Brownian parameter. Ghalambaz and colleagues investigated the impacts of hybrid nanoparticles on the melting process of a nanoenhanced phase-change material (NEPCM). Their results show that increasing the values of the nanoparticles volume fraction, viscosity and conductivity parameters leads to striking variations in the solid-liquid interface for large values of the Fourier number. Sheremet and colleagues examined the unsteady natural convection of water-based nanofluid within a wavy-walled cavity under the impact of a uniform inclined magnetic field. They found that a rise in the undulation number leads to an abatement in the average Nusselt number owing to more intensive heating of the wavy troughs, while an increase in the Hartmann number leads to a reduction of the decreasing rate of the average Nusselt number. Chamkha and colleagues explored unsteady conjugate natural convection in a semicircular cavity with a solid shell of finite thickness and filled with a hybrid water-based suspension of Al₂O₃ and Cu nanoparticles (hybrid nanofluid). Their outcomes illustrate that an addition of only 5% of Al₂O₃–Cu nanoparticles indicates an increase of $\frac{\text{Nu}}{\text{interface}}$ from 4.9 to 5.4 while an addition of 5% of Al₂O₃ nanoparticles leads to an increase of $\frac{\text{Nu}}{\text{interface}}$ from 4.9 to 5.36. Dogonchi and colleagues examined MHD Go-water nanofluid flow and heat transfer between two flat plates in the presence of thermal radiation. They stated that the temperature profile and the Nusselt number have a direct relationship with the solid volume fraction and an inverse relationship with the radiation parameter. The unsteady MHD nanofluid flow and heat transfer between the two infinite parallel plates with thermal radiation impact was analyzed by Dogonchi and colleagues. They reported that
the skin friction coefficient and Nusselt number ascend with a rise in the magnetic parameter and volume fraction of the nanofluid. Dogonchi and Ganji\textsuperscript{23} investigated MHD nanofluid flow, heat, and mass transfer between two nonparallel walls. They consider Brownian diffusion and thermophoresis effects, which are two momentous slip mechanisms in nanofluids. They realized that the temperature and concentration profiles and Nusselt number rise with an increasing Schmidt number. Further, they proved that with an ascending Brownian parameter or declining thermophoretic parameter, the concentration profile is augmented. Thermophysical properties of TiO$_2$–Cu–water nanofluid transport in MHD stagnation point flow considering shape factor was analyzed by Ghadikolaei and colleagues.\textsuperscript{24} Ghadikolaei and colleagues\textsuperscript{25} studied three-dimensional squeezing flow of mixture base fluid (ethylene glycol–water) suspended by hybrid nanoparticle (Fe$_3$O$_4$–Ag). Recently, nanofluid flow and heat transfer have been reviewed by many researchers.\textsuperscript{26–35}

The main goal of this work is to explore the flow and heat transfer of nanofluid between two parallel disks in the presence of thermal radiation and heat source impacts and considering the Cattaneo–Christov heat flux model instead of the more conventional Fourier’s law of heat conduction. The governing momentum and energy equations accompanied by the boundary conditions are first transmuted into a nondimensional form, and next they are solved by a new modification of the Adomian Decomposition Method (ADM),\textsuperscript{36} denominated as the Duan–Rach Approach (DRA).\textsuperscript{37} Unlike other modified methods of the ADM,\textsuperscript{38,39} this method allows us to determine a solution without using numerical methods to evaluate the unspecified coefficients. In truth, the final solution is not comprised of unspecified coefficients. This method was successfully applied to different engineering problems.\textsuperscript{40–42} The impacts of diverse active parameters, such as the suction/injection parameter, the solid volume fraction, the heat source parameter, the thermal relaxation parameter, and the radiation parameter on flow and heat transfer traits are examined.

2 | PROBLEM DESCRIPTION

Assume the unsteady two-dimensional, radiative squeezing nanofluid flow between two parallel disks in which one of the disks is penetrable and the other is stretchable/shrinkable (Figure 1). The Cattaneo–Christov heat flux model instead of the more conventional Fourier’s law of heat conduction is applied to explore the heat transfer traits. The two disks are placed at $h(t) = H(1-at)^{1/2}$. The upper disk at
Table 1  Thermophysical properties of water and nanoparticles

<table>
<thead>
<tr>
<th></th>
<th>(\rho (kg/m^3))</th>
<th>(C_p (J/kgK))</th>
<th>(k(W/mK))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper (Cu)</td>
<td>8933</td>
<td>385</td>
<td>401</td>
</tr>
<tr>
<td>Pure water</td>
<td>997.1</td>
<td>4179</td>
<td>0.613</td>
</tr>
</tbody>
</table>

\(z = h(t)\) departs towards or away from the stationary lower disk with the velocity \(dh/dt\). For \(a > 0\), the two disks are squeezed until they touch \(t = 1/a\) and for \(a < 0\) the two disks are separated. The nanofluid is taken to be incompressible and Newtonian. Also, it is presumed that both the nanoparticles and fluid phase are in a thermal equilibrium state and that they are remarkably small in size, so the slip velocity between the phases is neglected. The thermophysical properties of the nanofluid are given in Table 1.

Under the above presumptions, the governing equations for conservative momentum and energy in an unsteady two-dimensional nanofluid flow are:

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0, \tag{1}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_{nf}} \frac{\partial p}{\partial r} + \frac{\mu_{nf}}{\rho_{nf}} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right], \tag{2}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_{nf}} \frac{\partial p}{\partial z} + \frac{\mu_{nf}}{\rho_{nf}} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2} \right], \tag{3}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} + \epsilon \left[ \frac{\partial^2 T}{\partial r^2} + \frac{2u \partial^2 T}{\partial r \partial z} + \frac{2w \partial^2 T}{\partial z^2} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right] + \frac{\partial T}{\partial r} \left( u \frac{\partial u}{\partial r} + w \frac{\partial w}{\partial z} \right) + 2uw \frac{\partial^2 T}{\partial r \partial z} + u^2 \frac{\partial^2 T}{\partial r^2} + w^2 \frac{\partial^2 T}{\partial z^2} = \frac{k_{nf}}{(\rho C_p)_{nf}} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] - \frac{1}{(\rho C_p)_{nf}} \frac{\partial q_{rad}}{\partial z} + \frac{Q_0}{(\rho C_p)_{nf}} (T - T_H). \tag{4}
\]

Here \(u\) and \(w\) are the velocities in the \(r\) and \(z\) directions respectively, \(\epsilon\) is the relaxation time of heat flux, \(T\) is the temperature, \(P\) is the pressure and \(q_{rad.}\) is the radiative heat flux.

Using the Rosseland approximation for radiation we have:

\[
q_{rad.} = -\left( 4\sigma^* / 3k^*_{nf} \right) \frac{\partial T^4}{\partial z}, \tag{5}
\]

where \(\sigma^*\) is the Stefan–Boltzmann constant and \(k^*_{nf}\) is the mean absorption coefficient of the nanofluid.

Further, we suppose that the temperature difference within the flow is such that \(T^4\) may be expanded in a Taylor series. So, expanding \(T^4\) about \(T_\infty\) and neglecting higher-order terms, we obtain

\[
T^4 \approx 4T^3_\infty (T - 3T^3_\infty). \tag{6}
\]
Therefore, Eq. (4) is simplified to

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} + \varepsilon \left[ \frac{\partial^2 T}{\partial t^2} + 2u \frac{\partial^2 T}{\partial t \partial r} + 2w \frac{\partial^2 T}{\partial t \partial z} + \frac{\partial u}{\partial t} \frac{\partial T}{\partial r} + \frac{\partial w}{\partial t} \frac{\partial T}{\partial z} \right]
\]

\[
+ \frac{2uw}{r \partial r} \left( \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) + \frac{\partial u}{\partial z} \left( \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) + \frac{\partial T}{\partial r} \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) + \frac{\partial T}{\partial z} \left( u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right)
\]

\[
+ \frac{2}{r} \left( u \frac{\partial^2 T}{\partial t \partial r} + u \frac{\partial^2 T}{\partial r^2} + w \frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial r} \right)
\]

\[
= \frac{k_{nf}}{(\rho C_p)_{nf}} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{16 \sigma^2 T_0^3}{\rho C_v (\rho C_p)_{nf}} \frac{\partial^2 T}{\partial z^2} \right] + \frac{Q_0}{(\rho C_p)_{nf}} (T - T_H),
\]

where \( \rho_{nf} \) is the effective density of the nanofluid, \( \mu_{nf} \) is the effective dynamic viscosity of the nanofluid, \( (\rho C_p)_{nf} \) is the heat capacitance of the nanofluid, and \( k_{nf} \) is the thermal conductivity of the nanofluid are given as

\[
\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_s,
\]

\[
(\rho C_p)_{nf} = (1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_s,
\]

\[
\mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}},
\]

\[
k_{nf} = \frac{k_s + 2k_f - 2\phi (k_f - k_s)}{k_s + 2k_f + 2\phi (k_f - k_s)}.
\]

The relevant boundary conditions are:

\[
u = u_w = \frac{br}{2} (1 - at) \quad w = \frac{dh}{dt} \quad T = T_H \quad at \quad z = h(t)
\]

\[
u = 0 \quad w = -w_0 \sqrt{1 - at} \quad T = T_w \quad at \quad z = 0.
\]

We define these parameters:

\[
\eta = \frac{z}{H (1 - at)^{1/2}}, \quad u = \frac{ar}{2 (1 - at)^{1/2}} f'(\eta),
\]

\[
w = -\frac{aH}{(1 - at)^{1/2}} f(\eta), \quad \theta = \left( T - T_H \right) / \left( T_w - T_H \right).
\]

Substituting these into the governing equations and then eliminating the pressure gradient yields, the nonlinear fourth- and second-order ordinary differential equations for the momentum and energy equations, respectively:

\[
f'''' + \frac{A_1}{A_2} S \left( 2f''' - 3f'' - \eta f'''' \right) = 0,
\]

\[
\theta'' + \frac{A_3}{A_4} S \text{Pr} \left( \frac{3}{3+4N} \right) \left[ 2f' \theta' - \eta \theta' - \gamma \left( f' f' + f^2 \theta'' - \frac{\eta}{2} f' \theta' - \frac{\eta^2}{4} \theta'' - \eta f \theta'' \right) \right]
\]

\[
+ H_s \left( \frac{3}{3+4N} \right) \theta = 0.
\]

Here \( A_1, A_2, A_3, \) and \( A_4 \) are dimensionless constants defined by:

\[
A_1 = \frac{\rho_{nf}}{\rho_f}, \quad A_2 = \frac{\mu_{nf}}{\mu_f}, \quad A_3 = \frac{(\rho C_p)_{nf}}{(\rho C_p)_f}, \quad A_4 = \frac{k_{nf}}{k_f}.
\]
Subject to the boundary conditions

\[ f(0) = A, \quad f'(0) = 0, \quad f(1) = 1/2, \quad f''(1) = C, \quad \theta(0) = 1, \quad \theta(1) = 0. \]  

(14)

where \( S \) is the squeeze number, \( \gamma \) is the thermal relaxation parameter, \( \text{Pr} \) is the Prandtl number, \( Hs \) is the heat source parameter, \( C \) is the stretching \((C > 0)\) or shrinking \((C < 0)\) parameter, and \( A \) is the suction/injection parameter. It is necessary to mention that \( A > 0 \) shows the suction of fluid from the lower disk while \( A < 0 \) demonstrates injection flow.

Other significant characteristics of the present work are the skin friction coefficient and the Nusselt number, which are determined as follows:

\[ C_f^* = \frac{\mu_n f}{\rho f \sqrt{aH / 2 \sqrt{1 - at}}} \left. \frac{\partial u}{\partial z} \right|_{z=h(t)} \],

\[ Nu^* = - \left[ \frac{H}{k_f(T_w-T_h)} \right] \left( k_n f + \frac{16\sigma T^3}{3k_n f} \right) \left. \frac{\partial T}{\partial z} \right|_{z=h(t)}. \]  

(15)

In terms of (10) and (13), we gain:

\[ C_f = \left| A_2 f''(1) \right|, \]

\[ Nu = \left| A_4 \left( 1 + \frac{4}{3}N \right) \theta'(1) \right|. \]  

(16)

3 | IMPLEMENTATION OF THE DUAN–RACH APPROACH

Based on the descriptions provided in,37,40–42 the DRA must be modified. We do not use the prescribed value\( \xi \). Hence, according to DRA, Eqs. (11) and (12) can be written as follows:

\[ L_4 f(\eta) = -\frac{A_1}{A_2} S \left( 2f''' - 3f'' - \eta f''' \right), \]  

\[ L_2 \theta(\eta) = -\frac{A_1}{A_2} S \text{Pr} \left( \frac{3}{3+4N} \right) \left[ 2f \theta' - \eta \theta' - \gamma \left( f' \theta' + f^2 \theta'' - \frac{\eta}{2} f' \theta' - \frac{\eta^2}{4} \theta'' - \eta f \theta'' \right) \right] \]  

(17)

(18)

where the differential operator \( L_4 \) and \( L_2 \) are given by \( L_4 = \frac{d^4}{d\eta^4} \) and \( L_2 = \frac{d^2}{d\eta^2} \), respectively. Suppose that the inverse operator \( L_4^{-1} \) and \( L_2^{-1} \) exist, then we have:

\[ L_4^{-1}(\bullet) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (\bullet) d\eta d\eta d\eta d\eta, \quad L_2^{-1}(\bullet) = \int_0^\eta \int_0^\eta (\bullet) d\eta d\eta. \]  

(19)

Operating with \( L_4^{-1} \) in Eq. (17) and after exerting boundary conditions on it:

\[ f(\eta) = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2} + f'''(0)\frac{\eta^3}{6} + L_4^{-1}(N_1 u). \]  

(20)
Operating with $L_{2}^{-1}$ in Eq. (18) and after exerting boundary conditions on it:

$$\theta(\eta) = \theta(0) + \theta'(0)\eta + L_{2}^{-1}(N_{3}u),$$

where $N_{1}u, N_{2}u$ are defined as

$$N_{1}u = -\frac{A_{1}}{A_{2}}S \left(2ff''' - 3f'' - \eta f'''ight),$$

and

$$N_{2}u = -\frac{A_{3}}{A_{4}}S Pr \left(\frac{3}{3+4N}\right) \left[2f\theta' - \eta\theta' - \gamma \left(ff\theta' + f^{3}\theta'' - \frac{\eta}{2} f'\theta' - \frac{\eta^{2}}{4} \theta'' - \eta f\theta''\right)\right] - H_{s} \frac{4}{A_{4}} \left(\frac{3}{3+4N}\right) \theta.$$ (23)

Evidently, we do not have the values of $f''(0), f'''(0),$ and $\theta'(0)$. In the standard ADM, we need to evaluate those undetermined conditions with numerical methods. Hence, the boundary value problem (BVP) is turned into an initial value problem (IVP). The accuracy of the solution depends on the accuracy of the three undetermined parameters. In our work, we use the DRA$^{37}$ to detect a totally analytical solution.

Operating with $L_{4}^{-1}$ on Eq. (17) at $\eta = 1$, we have:

$$\int_{0}^{1} \int_{0}^{\eta} \int_{0}^{\eta} \int_{0}^{\eta} f_{iv}(\eta)d\eta d\eta d\eta = \left[L_{4}^{-1} N_{1}u\right]_{\eta=1},$$

where

$$\left[L_{4}^{-1} N_{1}u\right]_{\eta=1} = \int_{0}^{1} \int_{0}^{\eta} \int_{0}^{\eta} \int_{0}^{\eta} (N_{1}u)d\eta d\eta d\eta d\eta.$$ (25)

After integration of the left hand side, we have

$$\frac{1}{2} - \frac{1}{2} f''(0) - \frac{1}{6} f'''(0) - A = \left[L_{4}^{-1} N_{1}u\right]_{\eta=1}. $$ (26)

Operating with $L_{3}^{-1}$ on Eq. (17) at $\eta = 1$, we have:

$$\int_{0}^{1} \int_{0}^{\eta} \int_{0}^{\eta} f_{iv}(\eta)d\eta d\eta d\eta = \left[L_{3}^{-1} N_{1}u\right]_{\eta=1},$$

where

$$\left[L_{3}^{-1} N_{1}u\right]_{\eta=1} = \int_{0}^{1} \int_{0}^{\eta} \int_{0}^{\eta} (N_{1}u)d\eta d\eta d\eta.$$ (28)

After integration of the left hand side, we have

$$C - f''(0) - \frac{1}{2} f'''(0) = \left[L_{3}^{-1} N_{1}u\right]_{\eta=1}. $$ (29)

The subtraction of Eq. (26) from Eq. (29) gives us the relation of $f'''(0)$:

$$f'''(0) = 12A + 6C - 6 + 12\left[L_{4}^{-1} N_{1}u\right]_{\eta=1} - 6\left[L_{3}^{-1} N_{1}u\right]_{\eta=1}. $$ (30)
TABLE 2 Comparison between present and Hashmi and colleagues results for $N_u$

<table>
<thead>
<tr>
<th>$N_b$</th>
<th>Present results</th>
<th>Hashmi and colleagues results</th>
<th>Present results</th>
<th>Hashmi and colleagues results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>N = 0.1</td>
<td>0.52628539</td>
<td>0.5</td>
<td>1.17682118</td>
</tr>
<tr>
<td>0.5</td>
<td>0.63433253</td>
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<td>1</td>
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<td>0.95569955</td>
<td>0.52628540</td>
<td>2</td>
<td>2.17922793</td>
</tr>
</tbody>
</table>

and $f''(0)$ is:

$$f''(0) = 3 - 6A - 2C - 6\left[L_4^{-1}N_1u\right]_{\eta=1} + 2\left[L_3^{-1}N_1u\right]_{\eta=1}.\quad (31)$$

Substituting $f''(0)$ and $f'''(0)$ into Eq. (20) yields,

$$f(\eta) = A + \left(\frac{3}{2} - 3A - C\right)\eta^2 + (2A + C - 1)\eta^3 + (2\eta^3 - 3\eta^2)\left[L_4^{-1}N_1u\right]_{\eta=1} + \left(L_3^{-1}N_1u\right)_{\eta=1}.\quad (32)$$

Thus, the right-hand side of Eq. (32) does not comprise the unspecified parameters $f''(0)$ and $f'''(0)$. Eventually, we have the modified recursive scheme:

$$f_0(\eta) = A + \left(\frac{3}{2} - 3A - C\right)\eta^2 + (2A + C - 1)\eta^3,$n$$f_{n+1}(\eta) = (2\eta^3 - 3\eta^2)\left[L_4^{-1}A_n(\eta)\right]_{\eta=1} + (\eta^2 - \eta^3)\left[L_3^{-1}A_n(\eta)\right]_{\eta=1} + \left(L_4^{-1}A_n(\eta)\right)_{\eta=1}.\quad (33)$$

where $A_n(\eta)$ are the Adomian polynomials, which can be specified by the formula

$$A_n(\eta) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N\left(\sum_{i=0}^{n} \lambda^i F_i(\eta)\right)\right]_{\lambda=0} \quad (34)$$

that was first demonstrated by Adomian and Rach.43

Applying Eq. (34), we gain the terms of the Adomian polynomials and put them in Eq. (33), and we specify $f_n(\eta)$ as follows:

$$f_0(\eta) = A + \left(\frac{3}{2} - 3A - C\right)\eta^2 + (2A + C - 1)\eta^3$$

$$f_1(\eta) = -\frac{1}{840} \frac{A_1}{A_2} S \left[\begin{array}{c} 528A^2 + 192AC - 36C^2 \\ -150A - 12C - 57 \end{array}\right] \eta^2$$

$$-\frac{1}{840} \frac{A_1}{A_2} S \left[\begin{array}{c} -1248A^2 - 520AC + 52C^2 \\ +156A - 76C + 234 \end{array}\right] \eta^3 + -\ldots \quad (35)$$

The functions $f_2(\eta), f_3(\eta), \ldots$ can be specified in a similar way from Eq. (33). For convenience, we do not represent all terms of $f_n(\eta)$.
Utilizing \( f(\eta) = \sum_{n=0}^{\infty} f_n(\eta) = f_0(\eta) + f_1(\eta) + f_2(\eta) + \ldots \), thus

\[
f(\eta) = A + \left(\frac{3}{2} - 3A - C\right) \eta^2 + (2A + C - 1) \eta^3 - \frac{1}{840} \frac{A_1}{A_2} S \left(\begin{array}{c} 528A^2 + 192AC - 36C^2 \\ -150A - 12C - 57 \end{array}\right) \eta^2 \\
- \frac{1}{840} \frac{A_1}{A_2} S \left(\begin{array}{c} -1248A^2 - 520AC + 52C^2 \\ +156A - 76C + 234 \end{array}\right) \eta^3 - \ldots \tag{36}\]

According to Eq. (36), the accuracy increases by increasing the number of solution terms \((n)\). For \(\theta(\eta)\), we proceed in the same manner. We get the following recursive scheme:

\[
\begin{align*}
\theta_0(\eta) &= 1 - \eta, \\
\theta_{n+1}(\eta) &= -\eta \left[ L_2^{-1} N_2 u \right]_{\eta=1} + \left[ L_2^{-1} N_2 u \right].
\end{align*}
\tag{37}\]
FIGURE 3 Impact of the suction/injection parameter on velocity and temperature profiles for both stretching \((C = 1)\) and shrinking \((C = -1)\) cases [Color figure can be viewed at wileyonlinelibrary.com]

where \([L_2^{-1} N_2 u]_{\eta=1} = f_0^1 f_0^\eta (N_2 u) d\eta d\eta\).

Utilizing \(\theta(\eta) = \sum_{n=0}^{\infty} \theta_n(\eta) = \theta_0(\eta) + \theta_1(\eta) + \theta_2(\eta) + \ldots\), thus

\[
\theta_0(\eta) = 1 - \eta - \frac{1}{840} \frac{3}{3 + 4N} \frac{A_3}{A_4} \Pr S \left( 24A^2 \gamma + 54AC \gamma - 27C^2 \gamma - 192A \gamma + 29C \gamma \right) + \ldots \quad (38)
\]

4 RESULTS AND DISCUSSION

In this paper, the unsteady squeezing flow and heat transfer of nanofluid between two parallel disks in which one of the disks is penetrable and the other stretchable/shrinkable, in the presence of
FIGURE 4 Impacts of the stretching/shrinking and suction/injection parameters on Nusselt number [Color figure can be viewed at wileyonlinelibrary.com]

thermal radiation and heat source impacts, and considering the Cattaneo–Christov heat flux model instead of the more conventional Fourier’s law of heat conduction is examined via DRA. The impacts of the suction/injection parameter ($A$), the solid volume fraction ($\phi$), the heat source parameter ($H_s$), the thermal relaxation parameter ($\gamma$), and the radiation parameter ($N$) on flow and heat transfer characteristics are studied. The nanofluid thermophysical properties have been abridged in Table 1.8 To validate the present numerical solution, we compared our results with other work reported in literature.28 They are in very good agreement as is shown in Table 2. It is necessary to state that all figures are portrayed for the squeezing flow case ($S < 0$).

Figures 2 to 4 portray the impacts of the stretching/shrinking and suction/injection parameters on velocity and temperature profiles and Nusselt number. It is essential to state that the stretching/shrinking parameter depicts the movement of the walls. That is $C > 0$ explains the stretching walls, $C < 0$ explains the shrinking walls and $C = 0$ explains the stationary walls. Further, it should be noted that $A > 0$ shows the suction of fluid from the lower disk while $A < 0$ indicates injection flow. It is manifest that for the both suction and injection processes, the soaring values of the stretching parameter lead to detract both the velocity and temperature profiles while the deportment of the velocity and temperature profiles is completely contrariwise for the shrinking parameter. Moreover, it can be concluded that for the suction process, by escalating the value of the stretching parameter, the probability of occurrence of the backflow phenomenon intensifies while this probability of occurrence is almost zero in the shrinking case. This fact is completely vice versa for injection process. That means for the injection process, by raising the absolute value of the shrinking parameter, the probability of occurrence of the backflow phenomenon escalates while this probability of occurrence is almost zero in the stretching case. Also, it is perceived that for both the suction and injection processes, shrinking warms up the system by increasing the thickness of the thermal layer while stretching cools down the system by thinning the thermal boundary layer. Hence, where cooling is essential, wall stretching is advisable. Furthermore, it can be seen that in both stretching and shrinking cases, the velocity profile has a direct relationship
with the suction parameter, while it has an inverse relationship with the magnitude of the injection parameter. In addition, it is seen that for both the stretching and shrinking cases, an overheating process occurs owing to both suction and injection and raising the thermal layer thickness. Moreover, it is worth mentioning that changing the wall state from stretching to shrinking ($C = 1$ to $C = -1$) leads to a growth of the Nusselt number from almost 3.768421 to 4.980549 for the suction process ($A = 1$) (almost a 32% augmentation in the Nusselt number), while for the injection process ($A = -1$), it causes a growth in the Nusselt number from almost 1.750650 to 2.437032 (an almost 39% augmentation in the Nusselt number). That means the impact of changing the wall state from stretching to shrinking becomes more significant in the injection process. On the other hand, the Nusselt number has a direct relationship with the magnitude of the shrinking parameter and with both the suction and injection parameters, while it has an inverse relationship with the stretching parameter. Also, in both stretching and shrinking cases, the Nusselt number is higher for the suction process than the injection. For instance, in the stretching
case ($C = 1$), the value of Nusselt number is almost 5.0391 for the suction process ($A = 2$) while for the injection ($A = -2$) its value is almost 2.1140. That means, changing the system from injection process to suction ($A = -2$ to $A = 2$) in the stretching case leads to almost a 138% augmentation in the Nusselt number. These obtained results indicate the suction/injection process and stretchable wall can be used as a control element for fluid flow and heat transfer within the engineering and industrial systems.
Figures 5 and 6 portray the impact of the nanofluid volume fraction and radiation parameter on the temperature profile and Nusselt number. It can be deduced that for the injection process, the augmenting values of the volume fraction of the nanofluid and radiation parameter lead to an ascent in the temperature of the nanofluid (for both stretching and shrinking cases). But this treatment of the temperature profile is completely reversed for the suction process. Conforming to Eq. (16), multiplication of three terms, i.e. the thermal conductivity rate \( \frac{k_{nf}}{k_f} \), \( 1+(4/3)N \) and temperature gradient, will result in a nondimensional number which is named as the Nusselt number. By increasing the nanofluid volume fraction, the augmentation of the thermal conductivity rate will overcome the reduction in the temperature gradient in a given radiation parameter. Also, by raising the radiation parameter, the augmentation in \( 1+(4/3)N \) term will overcome the reduction of the temperature gradient in a given solid volume fraction. So the Nusselt number will increase with the augmentation of the nanofluid volume fraction and radiation parameter. This treatment of the Nusselt number with a variation in
the nanofluid volume fraction and radiation parameter is the same for both stretching/shrinking cases and suction/injection processes. In fact, for both stretching/shrinking cases and suction/injection processes, the Nusselt number has a direct relationship with the nanofluid volume fraction and radiation parameter.

Figures 7 and 8 portray the impacts of the thermal relaxation and heat source parameters on the temperature profile and Nusselt number. Due to the fact that the heat source parameter causes a heat generation in the system, it is envisaged that the temperature profile will augment with an increase
in the heat source parameter. This behaviour of the temperature profile with an alteration of the heat source parameter for both stretching/shrinking cases and suction/injection processes can be seen in Figure 7. On the other hand, it can be viewed that for the suction process, the temperature profile of nanofluid abates with a rise in the thermal relaxation parameter, but for the injection process, the trend of the temperature profile changes. Another significant point that can be concluded is that for the suction process, the Nusselt number is a reducing function of the thermal relaxation parameter, while for the injection process, this treatment of the Nusselt number is completely the opposite. It is necessary to mention that the Nusselt number always has a direct relationship with the heat source parameter. This is due to the fact that the temperature gradient ascends with ascending the heat source parameter. According to Eq. (16), since the Nusselt number has a direct relationship with the temperature gradient, so the Nusselt number ascends with ascending heat source parameter. Therefore, the presence of the heat source leads to the heating of the system. Hence, where the heating is essential, it is recommended to use a heat source in the system.

5 | CONCLUSIONS

In this work, the unsteady squeezing flow and heat transfer of a nanofluid between two parallel disks in which one of the disks is penetrable and the other is stretchable/shrinkable, in the presence of thermal radiation and heat source impacts, and considering the Cattaneo–Christov heat flux model instead of the more conventional Fourier's law of heat conduction is examined via the DRA. This method permits us to detect an analytical solution without applying a numerical method to compute the missing parameters $f''(0), f'''(0)$ and $\theta'(0)$. The comparison between DRA and other work reported in literature proves the validity of this method. The outcomes indicate that for both stretching/shrinking cases and suction/injection processes, the Nusselt number has a direct relationship with the nanofluid volume fraction and radiation parameter. The outcomes also reveal that the Nusselt number has a direct relationship with the magnitude of the shrinking parameter and with both the suction and injection parameters, while it has an inverse relationship with the stretching parameter. Moreover, for the suction process, the Nusselt number is a reducing function of the thermal relaxation parameter, while for the injection process, this treatment of the Nusselt number is completely the reverse.

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