HEAT AND MASS TRANSFER ON UNSTEADY, MAGNETOHYDRODYNAMIC, OSCILLATORY FLOW OF SECOND-GRADE FLUID THROUGH A POROUS MEDIUM BETWEEN TWO VERTICAL PLATES, UNDER THE INFLUENCE OF FLUCTUATING HEAT SOURCE/SINK, AND CHEMICAL REACTION

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We consider the unsteady, magnetohydrodynamic, oscillatory flow of an incompressible, electrically conducting, second-grade fluid through a saturated, porous medium between two vertical plates that are under the influence of a uniform, transverse, magnetic field normal to the plates, with heat source and chemical reaction. One plate of the vertical channel is kept stationary, whereas the other is oscillating with uniform velocity; the two plates are subjected to constant injection and suction velocities, respectively. The flow through the porous medium is governed by the equation for Brinkman’s model for momentum. The closed-form solutions of the governing equations are obtained for velocity, temperature, and concentration profiles, with use of the perturbation technique. The effects of various governing parameters on these three profiles are computationally discussed and graphically presented. Skin friction, Nusselt number, and Sherwood number are obtained analytically, and their behaviors are computationally discussed.

KEY WORDS: heat and mass transfer, vertical plates, MHD flows, oscillatory flows, porous medium, and second-grade fluids

1. INTRODUCTION

The phenomenon of heat and mass transfer is observed in buoyancy-induced motions in atmosphere, water bodies, quasisolid bodies such as Earth, etc. Oscillatory, free-convective flows have an important role in chemical engineering, turbo-machinery, and aerospace technology. The study of such flows was initiated by Lighthill (1954), who reviewed the effects of free oscillations on the flow of a viscous, incompressible fluid past an infinite plate. Nanda and Sharma (1963) extended the theory for free-convection boundary layers along a semi-infinite, vertical plate. Analytical solutions for heat and mass transfer by laminar flow of a Newtonian, viscous, electrically conducting, and heat-generating or -absorbing fluid on a continuously vertical, permeable surface in the presence of radiation, first-order homogeneous chemical reaction, and mass flux were reported by Ibrahim et al. (2008). Hussanan et al. (2014a) presented an exact
analysis of unsteady, natural convection flow of viscous fluid past an oscillating plate with Newtonian heating. The problem of magnetohydrodynamic (MHD) natural convection about a vertical impermeable flat plate was presented by Sparrow and Cess (1961), Wilks and Hunt (1984), and Cussler (1998). Fairbanks and Wike (1950) studied the effects of chemical reaction and diffusion in an isothermal, laminar flow along a soluble flat plate. Das et al. (1994) reported on the effects of chemical reaction and mass transfer on flow past an impulsively started, infinite, vertical plate with constant heat flux. Takhar et al. (2000) investigated flow and mass diffusion of chemical species with first- and higher-order, reactions over a continuously stretching sheet with magnetic-field effect. Muthucumaraswamy (2002) studied the effects of chemical reaction on a moving, isothermal, vertical, infinitely long surface with suction. Loganathan et al. (2008) investigated the effects of homogeneous first-order chemical reaction and mass diffusion on unsteady flow past an impulsively started, semi-infinite, vertical plate with variable temperature in the presence of thermal radiation.

Unsteady heat and mass transfer by free-convective MHD micropolar fluid flow with chemical reaction in the presence of heat generation was studied by Olajuwon and Oahimire (2014). Recently, finite-difference analysis of MHD, free-convective flow of an incompressible, viscous-dissipative fluid in an infinite, vertical, oscillating plate with constant heat flux has been performed (Gebhart, 1962; Pantokratoras, 2003; Kishan et al., 2006; Kishan and Amrutha, 2010; Kishan and Dessie, 2014). Hayat (2008c) discussed some MHD flows of second-grade fluid ($\alpha$) through a porous medium. Marques et al. (2000) considered the effect of fluid slippage at the plate for Couette flow. Hayat et al. (2008b) analyzed slip flow and heat transfer of $\alpha$ past a stretching sheet through a porous

Keeping the above-mentioned studies in mind, in this paper we consider unsteady, MHD oscillatory flow of incompressible, electrically conducting $\alpha$ through a saturated, porous medium between two vertical plates, under the influence of uniform, transverse magnetic field normal to the plates, with heat source and chemical reaction.  

2. MATHEMATICAL FORMULATION AND SOLUTION TO THE PROBLEM

We consider the flow of unsteady, hydromagnetic flow of an incompressible, electrically conducting, viscous $\alpha$ through a saturated, porous medium bounded by two insulated vertical plates that are distance $d$ apart in the presence of the heat source, chemical reaction, and uniform, homogeneous, magnetic field of strength $B_0$, normal to the plane of plates, as shown in Fig. 1.

We choose a Cartesian coordinate system $O(x, y, z)$, such that the boundary walls are at $z = 0$ and $z = d$, with origin at the stationary plate that is subjected to a constant injection velocity $w_0$. The other plate is oscillating in its own plane, at a velocity $U(t)$, about a constant mean velocity $U_0$, and subjected to the same constant, suction velocity.
With the above frame of reference and assumptions, that is, because the plates of the channel extend to infinity along the $x$ direction, all of the physical quantities except for pressure depend on $z$ and $t$ alone.

The unsteady, hydromagnetic flow of an incompressible, electrically conducting, viscous $\alpha$ through a saturated, porous medium is governed by the equation of motion for momentum, conservation of energy, and equation of mass transfer under the usual Boussinesq approximation, and is given by the following equations:

\[
\frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad z = w_0, \\
\frac{\partial u}{\partial t} + w_0 \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 u}{\partial z^3 \partial t} - \frac{\sigma B_0^2}{\rho} u - \frac{\nu}{K_1} u + g\beta(T - T_d) + g\beta^*(C - C_d),
\]

\[
\frac{\partial v}{\partial t} + w_0 \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 v}{\partial z^3 \partial t} - \frac{\sigma B_0^2}{\rho} v - \frac{\nu}{K_1} v,
\]

\[
\frac{\partial T}{\partial t} + w_0 \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial z^2} + \frac{Q}{\rho C_p} (T - T_d),
\]

\[
\frac{\partial C}{\partial t} + w_0 \frac{\partial C}{\partial z} = \frac{D}{\rho} \frac{\partial^2 C}{\partial z^2} + D_1 \frac{\partial^2 C}{\partial z^2} - R(C - C_d).
\]

Combining Eqs. (2) and (3), let $q = u + iv$ and $\xi = x - iy$, and we obtain

\[
\frac{\partial q}{\partial t} + w_0 \frac{\partial q}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial \xi} + \nu \frac{\partial^2 q}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 q}{\partial z^3 \partial t} - \frac{\sigma B_0^2}{\rho} q - \frac{\nu}{K_1} q + g\beta(T - T_d) + g\beta^*(C - C_d).
\]

The corresponding boundary conditions are

\[
q = 0, \quad T = T_0 + (T_0 - T_d) e^{i\omega t}, \quad C = C_0 + (C_0 - C_d) e^{i\omega t} \quad \text{at} \quad z = 0,
\]

\[
q = U(t) = U_0(1 + e^{i\omega t}), \quad T = T_d, \quad C = C_d \quad \text{at} \quad y = d.
\]

Eliminating the modified pressure gradient under the usual boundary-layer approximation, Eq. (2) reduces to

\[
\frac{\partial q}{\partial t} + w_0 \frac{\partial q}{\partial z} = \frac{\partial U}{\partial t} + \nu \frac{\partial^2 q}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 q}{\partial z^3 \partial t} - \frac{\sigma B_0^2 (q - U)}{\rho} - \frac{\nu(q - U)}{K_1} + g\beta(T - T_d) + g\beta^*(C - C_d),
\]

**FIG. 1:** Physical configuration of the problem
introducing the following nondimensional quantities:

\[ z^* = \frac{z}{d}, \quad u^* = \frac{u}{U_0}, \quad U^* = \frac{U}{U_0}, \quad \theta^* = \frac{T - T_d}{T_0 - T_d}, \quad \phi^* = \frac{C - C_d}{C_0 - C_d}, \quad \omega^* = \frac{\omega d}{w_0}, \quad t^* = \frac{tw_0}{d}. \]

Making use of nondimensional quantities (dropping asterisks), governing Eqs. (9), (4), and (5) can be written as

\[
\frac{\partial q}{\partial t} + \frac{\partial q}{\partial z} = \frac{\partial U}{\partial t} + \frac{1}{Re} \frac{\partial^2 q}{\partial z^2} + \left( \frac{M^2}{Re} + \frac{1}{K} \right) (q - U) + \text{Gr Re} \theta + \text{Gm Re} \phi, \tag{10}
\]

\[
\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial z} = \frac{1}{Pe} \frac{\partial^2 \theta}{\partial z^2} + S \theta, \tag{11}
\]

\[
\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial z} = \frac{1}{Sc} \frac{\partial^2 \phi}{\partial z^2} + S_o \frac{\partial^2 \theta}{\partial z^2} - \text{R} \phi, \tag{12}
\]

with the following boundary conditions:

\[ q = 0, \quad \theta = 1 + \frac{\varepsilon}{2} (e^{i\omega t} + e^{-i\omega t}), \quad \phi = 1 + \frac{\varepsilon}{2} (e^{i\omega t} + e^{-i\omega t}), \quad \text{at} \quad z = 0, \tag{13}\]

\[ q = 1 + \frac{\varepsilon}{2} (e^{i\omega t} + e^{-i\omega t}), \quad \theta = 0, \quad \phi = 0, \quad \text{at} \quad q = 1, \tag{14}\]

where

\[ M^2 = \frac{\sigma B_0^2 d^2}{\mu} \]

is the Hartmann number (magnetic-field parameter),

\[ K = \frac{K_1 w_0}{\nu d} \]

is the permeability parameter,

\[ Re = \frac{w_0 d}{\nu} \]

is the Reynolds number,

\[ \alpha = \frac{\alpha_1}{\rho d^2} \]

is the \( \alpha \) parameter,

\[ \text{Gr} = \frac{g^2 \nu (T_0 - T_d)}{U_0 w_0^2} \]

is the thermal Grashof number,

\[ \text{Gm} = \frac{g^2 \nu (C_0 - C_d)}{U_0 w_0^2} \]

is the mass Grashof number,

\[ \text{Pe} = \frac{\rho C_p U_0 w_0 d}{k} \]

is the Peclet parameter,

\[ S = \frac{Q d}{\rho C_p w_0} \]

is the heat source parameter,

\[ S_o = \frac{D_1 (T_0 - T_d)}{d w_0 (C_0 - C_d)} \]

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is the Soret parameter, 
\[ R = \frac{Rd}{w_0^2} \]
is the chemical-reaction parameter, and
\[ Sc = \frac{\nu}{D} \]
is the Schmidt number.

To solve Eqs. (10)–(12) for purely oscillatory flow, we assume the solution, following (Hamza, 1964):
\[ q = q_0(z) + \frac{\xi}{2} q_1(z) e^{i\omega t} + \frac{\xi}{2} q_2(z) e^{-i\omega t}, \]  
\[ \theta = \theta_0(z) + \frac{\xi}{2} \theta_1(z) e^{i\omega t} + \frac{\xi}{2} \theta_2(z) e^{-i\omega t}, \]  
\[ \phi = \phi_0(z) + \frac{\xi}{2} \phi_1(z) e^{i\omega t} + \frac{\xi}{2} \phi_2(z) e^{-i\omega t}, \]  
\[ U = U_0 \left( 1 + \frac{\xi}{2} e^{i\omega t} + \frac{\xi}{2} e^{-i\omega t} \right). \]  

Using Eqs. (15)–(18) into (10)–(12), and equating the coefficient of like terms, we get the following set of equations:
\[ \frac{d^2 \theta_0}{dz^2} - Pe \frac{d \theta_0}{dz} + SPe \theta_0 = 0, \]  
\[ \frac{d^2 \theta_1}{dz^2} - Pe \frac{d \theta_1}{dz} + (S - i\omega) Pe \theta_1 = 0, \]  
\[ \frac{d^2 \theta_2}{dz^2} - Pe \frac{d \theta_2}{dz} + (S + i\omega) Pe \theta_2 = 0, \]  
\[ \frac{d^2 \phi_0}{dz^2} - Sc Pe \frac{d \phi_0}{dz} - Sc Pe R \phi_0 = -S_0 Sc Pe \frac{d^2 \theta_0}{dz^2}, \]  
\[ \frac{d^2 \phi_1}{dz^2} - Sc Pe \frac{d \phi_1}{dz} - Sc Re (R + i\omega) \phi_1 = -S_0 Sc Re \frac{d^2 \theta_1}{dz^2}, \]  
\[ \frac{d^2 \phi_2}{dz^2} - Sc Pe \frac{d \phi_2}{dz} - Sc Re (R - i\omega) \phi_2 = -S_0 Sc Re \frac{d^2 \theta_2}{dz^2}, \]  
\[ \frac{d^2 q_0}{dz^2} - Re \frac{d q_0}{dz} - \left( M^2 + \frac{Re}{K} \right) q_0 = -Gr Re^2 \theta_0 - Gm Re^2 \phi_0 - \left( M^2 + \frac{Re}{K} \right) U_0, \]  
\[ (1 + \alpha i \omega Re) \frac{d^2 q_1}{dz^2} - Re \frac{d q_1}{dz} - \left( M^2 + \frac{Re}{K} + i\omega Re \right) q_1 = -Gr Re^2 \theta_1 - Gm Re^2 \phi_1 - \left( M^2 + \frac{Re}{K} + i\omega Re \right) U_0, \]  
\[ (1 - \alpha i \omega Re) \frac{d^2 q_2}{dz} - Re \frac{d q_2}{dz} - \left( M^2 + \frac{Re}{K} - i\omega Re \right) q_2 = -Gr Re^2 \theta_2 - Gm Re^2 \phi_2 - \left( M^2 + \frac{Re}{K} - i\omega Re \right) U_0. \]  

Corresponding boundary conditions are
\[ u_0 = u_1 = u_2 = 0, \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \phi_0 = \phi_1 = \phi_2 = 1, \quad \text{at} \quad z = 0, \]  
\[ u_0 = u_1 = u_2 = 1, \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \phi_0 = \phi_1 = \phi_2 = 0, \quad \text{at} \quad z = 1. \]

Solving Eqs. (19)–(27) using boundary conditions of Eqs. (28) and (29), we obtain the temperature, concentration, and velocity fields, given by
\[ \theta = a_1 e^{m_1 z} + a_2 e^{m_2 z} + \frac{\xi}{2} (a_3 e^{m_1 z} + a_4 e^{m_2 z}) e^{i\omega t} + \frac{\xi}{2} (a_5 e^{m_1 z} + a_6 e^{m_2 z}) e^{-i\omega t}, \]
\[
\phi = a_9 e^{m_{12}z} + a_{10} e^{m_{12}z} + a_7 e^{m_{12}z} + a_8 e^{m_{12}z} + \frac{\varepsilon}{2} (a_{13} e^{m_{12}z} + a_{14} e^{m_{12}z} + a_{11} e^{m_{12}z} + a_{12} e^{m_{12}z}) e^{i \omega t} \\
+ \frac{\varepsilon}{2} (a_{17} e^{m_{12}z} + a_{18} e^{m_{12}z} + a_{15} e^{m_{12}z} + a_{16} e^{m_{12}z}) e^{-i \omega t}.
\]
\]

\[
u = a_{27} e^{m_{12}z} + a_{28} e^{m_{12}z} + a_{19} e^{m_{12}z} + a_{20} e^{m_{12}z} + a_{21} e^{m_{12}z} + a_{22} e^{m_{12}z} + a_{23} e^{m_{12}z} + a_{24} e^{m_{12}z} + U_0 \\
+ \frac{\varepsilon}{2} (a_{37} e^{m_{12}z} + a_{38} e^{m_{12}z} + a_{29} e^{m_{12}z} + a_{30} e^{m_{12}z} + a_{31} e^{m_{12}z} + a_{32} e^{m_{12}z} + a_{33} e^{m_{12}z} + a_{34} e^{m_{12}z} + U_0) e^{i \omega t} \\
+ \frac{\varepsilon}{2} (a_{47} e^{m_{12}z} + a_{48} e^{m_{12}z} + a_{49} e^{m_{12}z} + a_{50} e^{m_{12}z} + a_{51} e^{m_{12}z} + a_{52} e^{m_{12}z} + a_{53} e^{m_{12}z} + a_{54} e^{m_{12}z} + U_0) e^{-i \omega t}.
\]

The nondimensional skin friction at the moving plate of the channel is given by

\[
\tau = -\mu \left( \frac{\partial \nu}{\partial z} \right)_{z=1} = a_{27} m_{13} e^{m_{12}z} + a_{28} m_{14} e^{m_{12}z} + a_{19} m_1 e^{m_{12}z} + a_{20} m_2 e^{m_{12}z} + a_{21} m_7 e^{m_{12}z} + a_{22} m_8 e^{m_{12}z} \\
+ a_{23} m_1 e^{m_{12}z} + a_{24} m_2 e^{m_{12}z} + \frac{\varepsilon}{2} (a_{37} m_{15} e^{m_{12}z} + a_{38} m_{16} e^{m_{12}z} + a_{29} m_3 e^{m_{12}z} + a_{30} m_4 e^{m_{12}z} + a_{31} m_9 e^{m_{12}z} \\
+ a_{32} m_5 e^{m_{12}z} + a_{33} m_6 e^{m_{12}z} + a_{34} m_4 e^{m_{12}z}) e^{i \omega t} + \frac{\varepsilon}{2} (a_{47} m_{17} e^{m_{12}z} + a_{48} m_{18} e^{m_{12}z} + a_{39} m_5 e^{m_{12}z} + a_{40} m_6 e^{m_{12}z} + a_{41} m_11 e^{m_{12}z} + a_{42} m_{12} e^{m_{12}z} + a_{43} m_5 e^{m_{12}z} + a_{44} m_6 e^{m_{12}z}) e^{-i \omega t}.
\]

The rate of heat transfer coefficient at the moving plate of the channel in terms of amplitude and phase angle is given by

\[
Nu = -\left( \frac{\partial T}{\partial z} \right)_{z=1} = a_1 m_1 e^{m_{12}z} + a_2 m_2 e^{m_{12}z} + \frac{\varepsilon}{2} (a_3 m_1 e^{m_{12}z} + a_4 m_4 e^{m_{12}z}) e^{i \omega t} \\
+ \frac{\varepsilon}{2} (a_5 m_5 e^{m_{12}z} + a_6 m_6 e^{m_{12}z}) e^{-i \omega t}.
\]

The rate of mass transfer coefficient at the moving plate of the channel in terms of amplitude and phase angle is given by

\[
Sh = \left( \frac{\partial C}{\partial z} \right)_{z=1} = a_9 m_7 e^{m_{12}z} + a_{10} m_8 e^{m_{12}z} + a_7 m_1 e^{m_{12}z} + a_8 m_2 e^{m_{12}z} + \frac{\varepsilon}{2} (a_{13} m_9 e^{m_{12}z} + a_{14} m_{10} e^{m_{12}z} \\
+ a_{11} m_3 e^{m_{12}z} + a_{12} m_4 e^{m_{12}z}) e^{i \omega t} + \frac{\varepsilon}{2} (a_{17} m_{11} e^{m_{12}z} + a_{18} m_{12} e^{m_{12}z} + a_{15} m_5 e^{m_{12}z} + a_{16} m_6 e^{m_{12}z}) e^{-i \omega t}.
\]

## 3. RESULTS AND DISCUSSION

In this article, we present the unsteady, MHD, free-convection flow of an incompressible, electrically conducting \( \alpha \) bounded by saturated, porous medium through two vertical plates in the presence of heat source and chemical reaction. The closed-form solutions for velocity, temperature, and concentration are obtained using perturbation technique. Velocity expression consists of the steady state and oscillatory state, revealing that the steady part of the velocity field has three-layer characters, whereas the oscillatory portion of the fluid field exhibits a multilayer character. Figures 2–12 show the effects of nondimensional parameters, including \( M \) the Hartmann number (magnetic-field parameter \( M \)), permeability parameter \( K \), \( \alpha \) thermal Grashof number \( Gr \), mass Grashof number \( Gm \), Peclet parameter \( Pe \), heat source parameter \( S \), Soret parameter \( S_o \), chemical reaction parameter \( R \), and Schmidt number \( Sc \). Figure 13 exhibits temperature distribution with different variations in governing parameters \( S, Pe, \omega t, \) and time \( t \), whereas the fixed parameters are \( U_0 = 1, Re = 1, \varepsilon = 0.001 \).

We noticed in Figs. 2, 9, and 12 that the magnitudes of components \( u \) and \( v \) reduce with increasing \( M, S_o \), or \( \omega t \). The application of transverse magnetic field has the important role of a resistive-type force (Lorentz force) that is similar to drag force (acts in the opposite direction of fluid motion) and tends to resist the flow, thereby reducing its velocity. Resultant velocity also decreases with increasing \( M, S_o \), or \( \omega t \). Figures 3, 6–8, and 10 show that both
FIG. 2: Velocity profiles for $u$ and $v$ against $M = 1$, $Gr = 3$, $Gm = 5$, $Pe = 2$, $Sc_{\alpha} = 0.2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $R = 1$, $t = 1$

FIG. 3: Velocity profiles for $u$ and $v$ against $K = 2$, $Gr = 3$, $Gm = 5$, $Pe = 2$, $Sc_{\alpha} = 0.2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $R = 1$, $t = 1$

FIG. 4: Velocity profiles for $u$ and $v$ against $Gr = 2$, $K = 1$, $Gm = 5$, $Pe = 2$, $Sc_{\alpha} = 0.2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $R = 1$, $t = 1$
FIG. 5: Velocity profiles for $u$ and $v$ against $Gm = 2$, $K = 1$, $Gr = 3$, $Pe = 2$, $S_o = 0.2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $R = 1$, $t = 1$

FIG. 6: Velocity profiles for $u$ and $v$ against $Pe = 2$, $K = 1$, $Gr = 3$, $Gm = 5$, $S_o = 0.2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $R = 1$, $t = 1$

FIG. 7: The velocity profiles for $u$ and $v$ against $R = 1$, $K = 1$, $Gr = 3$, $Gm = 5$, $Pe = 2$, $S_o = 0.2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $t = 1$
FIG. 8: Velocity profiles for $u$ and $v$ against $S M = 2$, $K = 1$, $Gr = 3$, $Gm = 5$, $Pe = 2$, $S_o = 0.2$, $\alpha = 1$, $Sc = 0.22$, $\omega t = \pi/2$, $R = 1$, $t = 1$

FIG. 9: Velocity profiles for $u$ and $v$ against $S_o M = 2$, $K = 1$, $Gr = 3$, $Gm = 5$, $Pe = 2$, $S = 1$, $\alpha = 1$, $Sc = 0.22$, $\omega = \pi/2$, $R = 1$, $t = 1$

FIG. 10: Velocity profiles for $u$ and $v$ against $Sc M = 2$, $K = 1$, $Gr = 3$, $Gm = 5$, $Pe = 2$, $S_o = 0.2$, $S = 1$, $\alpha = 1$, $\omega t = \pi/2$, $R = 1$, $t = 1$
FIG. 11: Velocity profiles for $u$ and $v$ against $\alpha M = 2, K = 1, Gr = 3, Gm = 5, Pe = 2, S_o = 0.2, S = 1, Sc = 0.22, \omega t = \pi/2, R = 1, t = 1$

FIG. 12: Velocity profiles for $u$ and $v$ against $\omega t M = 2, K = 1, Gr = 3, Gm = 5, Pe = 2, S_o = 0.2, S = 1, \alpha = 1, Sc = 0.22, R = 1, t = 1$

FIG. 13.
FIG. 13: Temperature profiles for $\theta$

FIG. 14: Concentration profiles for $\phi$
$u$ and $v$ enhance with increasing $K$, $Pe$, $R$, and $Sc$. Similar behavior is observed for resultant velocity. We observe that the lower the permeability of the porous medium, the lower the fluid speed in the entire fluid region. Increasing $Gr$, $Gm$, or $\alpha$ leads to enhanced primary velocity $u$ and reduced secondary velocity $v$ throughout the fluid region. Resultant velocity increases with increasing $Gr$, $Gm$, or $\alpha$ (Figs. 4, 5, and 11).

Temperature profiles are exhibited in Fig. 13 for different variations in $S$, $Pe$, $\omega t$, and $t$. Temperature increases in all layers with increasing $S$. We observe that $Pe$ leads to increased temperature uniformly in all layers, because $S$ is fixed. We conclude that $S$ and Prandtl number $Pr$ augment temperature in all layers. Temperature decreases with increasing $\omega t$ and increases with $t$.

Concentration profiles are shown in Fig. 14 for different variations in Schmidt number $Sc$, $R$, $S_o$, $\omega t$, and $t$. We note that concentration decreases at all layers of the flow for heavier species such as $CO_2$, $H_2O$, and $NH_3$, having Schmidt numbers 0.3, 0.6, and 0.78, respectively. We observe that for heavier-diffusing foreign species, velocity reduces with increasing $Sc$ in both magnitude and extent and thinning of thermal boundary layer. Likewise, concentration profiles increase with increasing $R$. We conclude that $Sc$ reduces and $R$ increases concentration in all layers. Concentration increases with increasing $S_o$ and $\omega t$ and reduces with $t$.

It can be seen in Table 1 that magnitudes of both skin-friction ($\tau$) components $\tau_x$ and $\tau_y$ and stress $|\tau|$ decrease with $M$ and $R$ an increase with $K$, $Gm$, and $S_o$. $\tau_x$ Increases, $\tau_y$ is stationary, and $|\tau|$ initially decreases and then

| $M$ | $K$ | $Gr$ | $Gm$ | $Pe$ | $R$ | $S$ | $S_o$ | $Sc$ | $\alpha$ | $\omega t$ | $\frac{\pi}{2}$ | $\tau_x$ | $\tau_y$ | $|\tau|$ |
|-----|-----|------|------|------|-----|-----|------|------|-------|---------|---------|--------|--------|--------|
| 2   | 0.5 | 3    | 5    | 2    | 1   | 1   | 2    | 0.22 | 1     | $\pi/2$  | 0.278240 | 0.421749 | 0.505263 |
| 2.5 |     |      |      |      |     |     |      |      |       |         | 0.049244 | 0.305504 | 0.309448 |
| 3   |     |      |      |      |     |     |      |      |       |         | -0.08775 | 0.220589 | 0.237402 |
| 1   |     |      |      |      |     |     |      |      |       |         | 0.451222 | 0.496358 | 0.670800 |
| 1.5 |     |      |      |      |     |     |      |      |       |         | 0.525245 | 0.525455 | 0.742957 |
| 4   |     |      |      |      |     |     |      |      |       |         | 0.038536 | 0.421749 | 0.424023 |
| 5   |     |      |      |      |     |     |      |      |       |         | -0.19053 | 0.421749 | 0.462791 |
| 6   |     |      |      |      |     |     |      |      |       |         | 0.338131 | 0.506099 | 0.608662 |
| 7   |     |      |      |      |     |     |      |      |       |         | 0.398021 | 0.590449 | 0.712075 |
| 2.5 |     |      |      |      |     |     |      |      |       |         | 0.315819 | 0.391808 | 0.503245 |
| 3   |     |      |      |      |     |     |      |      |       |         | 0.335100 | 0.442327 | 0.479072 |
| 2   |     |      |      |      |     |     |      |      |       |         | 0.182299 | 0.303562 | 0.354094 |
| 3   |     |      |      |      |     |     |      |      |       |         | 0.148593 | 0.248767 | 0.289767 |
| 2   |     |      |      |      |     |     |      |      |       |         | 0.306535 | 0.700050 | 0.764221 |
| 3   |     |      |      |      |     |     |      |      |       |         | 0.249850 | 0.889268 | 0.923701 |
| 1   |     |      |      |      |     |     |      |      |       |         | 0.089723 | 0.351369 | 0.362640 |
| 0.2 |     |      |      |      |     |     |      |      |       |         | -0.06109 | 0.295065 | 0.301323 |
| 0.3 |     |      |      |      |     |     |      |      |       |         | 0.317604 | 0.429406 | 0.534099 |
| 0.6 |     |      |      |      |     |     |      |      |       |         | 0.520141 | 0.547200 | 0.754966 |
| 2   |     |      |      |      |     |     |      |      |       |         | 0.278639 | 0.420260 | 0.504239 |
| 3   |     |      |      |      |     |     |      |      |       |         | 0.278552 | 0.419844 | 0.503846 |
| $\pi/3$ |     |      |      |      |     |     |      |      |       |         | 0.276255 | 0.422942 | 0.505154 |
| $\pi/4$ |     |      |      |      |     |     |      |      |       |         | 0.275106 | 0.422730 | 0.504365 |

$\alpha$, Second-grade fluid parameter; $Gr$, thermal Grashof number; $Gm$, mass Grashof number; $M$, magnetic parameter; $K$, porosity; $\omega t$, frequency of oscillation and time; $Pe$, Peclet parameter; $R$, chemical reaction; $S$, heat source parameter; $Sc$, Schmidt number; $S_o$, Soret parameter; $\tau$, skin friction ($\tau_x, \tau_y$); $|\tau|$, magnitude of stress.

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increases with increasing Gr. The magnitude of \( \tau_x \) increases and \( |\tau| \) decrease with increasing Pe. The magnitude of \( \tau_x \) increases, \( \tau_y \) reduces, and \( |\tau| \) increases with increasing Sc. \( \tau_x \) first increases and then reduced with increasing \( S, \alpha \), and \( \tau_y, \) and \( |\tau| \) increases with \( S \) and reduces with \( \alpha \). The magnitude of \( \tau_x \) and \( \tau_y \) enhance initially and then gradually reduce, and \( |\tau| \) decreases with increasing \( \omega t \).

In Table 2, the magnitude of the Nusselt number \( \text{Nu} \) increases for parameters \( S, \text{Pe}, \) and \( \omega \), but \( \text{Nu} \) reduces with \( t \). In Table 3, we can see that the magnitude of the Sherwood number \( \text{Sh} \) increases with \( t \) and reduces with increases in \( \text{Sc, } R, \text{So, } \) and \( \omega \).

### 4. CONCLUSIONS

We have considered the unsteady, MHD, oscillatory flow of incompressible, electrically conducting \( \alpha \) through a saturated, porous medium between two vertical plates under the influence of uniform, transverse, magnetic field normal to the plates, with heat source and chemical reaction. The following conclusions can be made:

**TABLE 2:** Nusselt number (\( \text{Nu} \))

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \text{Pe} )</th>
<th>( \omega t )</th>
<th>( t )</th>
<th>( \text{Nu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>( \pi/2 )</td>
<td>1</td>
<td>3.23197</td>
</tr>
<tr>
<td>2</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>4.77267</td>
</tr>
<tr>
<td>3</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>7.73040</td>
</tr>
<tr>
<td>—</td>
<td>3</td>
<td>—</td>
<td>—</td>
<td>4.10421</td>
</tr>
<tr>
<td>—</td>
<td>4</td>
<td>—</td>
<td>—</td>
<td>5.09829</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>( \pi/3 )</td>
<td>—</td>
<td>3.23284</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>( \pi/4 )</td>
<td>—</td>
<td>3.23317</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>—</td>
<td>1.5</td>
<td>3.22968</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>—</td>
<td>2</td>
<td>3.22781</td>
</tr>
</tbody>
</table>

\( \text{Nu} \), Nusselt number; \( \omega t \), frequency of oscillation and time; \( \text{Pe} \), Peclet parameter; \( S \), heat source parameter; \( t \), time.

**TABLE 3:** Sherwood number (\( \text{Sh} \))

<table>
<thead>
<tr>
<th>( \text{Sc} )</th>
<th>( R )</th>
<th>( \text{So} )</th>
<th>( \omega t )</th>
<th>( t )</th>
<th>( \text{Sh} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.22</td>
<td>1</td>
<td>0.2</td>
<td>( \pi/2 )</td>
<td>1</td>
<td>0.982618</td>
</tr>
<tr>
<td>0.3</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.980053</td>
</tr>
<tr>
<td>0.6</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.975455</td>
</tr>
<tr>
<td>—</td>
<td>2</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.945438</td>
</tr>
<tr>
<td>—</td>
<td>3</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.910510</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>0.4</td>
<td>—</td>
<td>—</td>
<td>0.892086</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>0.6</td>
<td>—</td>
<td>—</td>
<td>0.801586</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>—</td>
<td>( \pi/3 )</td>
<td>—</td>
<td>0.982331</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>—</td>
<td>( \pi/4 )</td>
<td>—</td>
<td>0.982795</td>
</tr>
<tr>
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<td>—</td>
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<td>—</td>
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<tr>
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<td>—</td>
<td>—</td>
<td>—</td>
<td>2</td>
<td>0.984666</td>
</tr>
</tbody>
</table>

\( \omega t \), frequency of oscillation and time; \( R \), chemical reaction parameter; \( \text{Sc} \), Schmidt number; \( \text{So} \), Soret parameter; \( \text{Sh} \), Sherwood number; \( t \), time.
1. Resultant velocity enhances with increasing $M$, $S_o$, and $\omega$ and decreases with $K$, $Gr$, $Gm$, $Pe$, $R$, $S$, $Sc$, and $\alpha$.

2. Temperature improves with $S$, $Pe$, and $t$ and reduces with increasing $\omega$.

3. $Sc$ and $t$ reduce concentration profiles and enhance with $R$, $S_o$, and $\omega$.

4. $|\tau|$ Reduces with increasing intensity of magnetic field and $Pe$ and increases with permeability of the porous medium $S_o$.

5. Nu augments with increasing $S$, $Pe$, and $\omega$.


REFERENCES


Sahin, A., Joaquin, Z., and Lopez-Ochoa, L.M., Numerical Modelling of MHD Convective Heat and Mass Transfer in Presence of


**APPENDIX A**

\[
\begin{align*}
m_1 &= \frac{\text{Pe} + \sqrt{\text{Pe}^2 - 4 \text{S} \text{Pe}}}{2}, \\
m_2 &= \frac{\text{Pe} - \sqrt{\text{Pe}^2 - 4 \text{S} \text{Pe}}}{2}, \\
m_3 &= \frac{\text{Pe} + \sqrt{\text{Pe}^2 - 4 (S - i\omega) \text{Pe}}}{2}, \\
m_4 &= \frac{\text{Pe} - \sqrt{\text{Pe}^2 - 4 (S - i\omega) \text{Pe}}}{2}, \\
m_5 &= \frac{\text{Pe} + \sqrt{\text{Pe}^2 - 4 (S + i\omega) \text{Pe}}}{2}, \\
m_6 &= \frac{\text{Pe} - \sqrt{\text{Pe}^2 - 4 (S + i\omega) \text{Pe}}}{2}, \\
m_7 &= \frac{\text{ScRe} + \sqrt{\text{Sc}^2 \text{Re}^2 - 4 \text{ScRe} R}}{2}, \\
m_8 &= \frac{\text{ScRe} - \sqrt{\text{Sc}^2 \text{Re}^2 - 4 \text{ScRe} R}}{2}, \\
m_9 &= \frac{\text{ScRe} + \sqrt{\text{Sc}^2 \text{Re}^2 + 4 \text{ScRe} (R + i\omega)}}{2}, \\
m_{10} &= \frac{\text{ScRe} - \sqrt{\text{Sc}^2 \text{Re}^2 + 4 \text{ScRe} (R + i\omega)}}{2},
\end{align*}
\]
\[ m_{11} = \frac{\text{ScRe} + \sqrt{\text{ScRe}^2 + 4\text{ScRe}(R - i\omega)}}{2}, \quad m_{12} = \frac{\text{ScRe} - \sqrt{\text{ScRe}^2 + 4\text{ScRe}(R - i\omega)}}{2}, \]
\[ m_{13} = \frac{\text{Re} + \sqrt{\text{Re}^2 + 4(M^2 + \text{Re}/K)}}{2}, \quad m_{14} = \frac{\text{Re} - \sqrt{\text{Re}^2 + 4(M^2 + \text{Re}/K)}}{2}, \]
\[ m_{15} = \frac{\text{Re} + \sqrt{\text{Re}^2 + 4(1 + \alpha\omega\text{Re})(M^2 + \text{Re}/K + i\omega\text{Re})}}{2}, \quad m_{16} = \frac{\text{Re} - \sqrt{\text{Re}^2 + 4(1 + \alpha\omega\text{Re})(M^2 + \text{Re}/K + i\omega\text{Re})}}{2}, \]
\[ m_{17} = \frac{\text{Re} + \sqrt{\text{Re}^2 + 4(1 - \alpha\omega\text{Re})(M^2 + \text{Re}/K - i\omega\text{Re})}}{2}, \quad m_{18} = \frac{\text{Re} - \sqrt{\text{Re}^2 + 4(1 - \alpha\omega\text{Re})(M^2 + \text{Re}/K - i\omega\text{Re})}}{2}. \]

\[ a_1 = \frac{-e^{m_3}}{e^{m_1} - e^{m_2}}, \quad a_2 = \frac{e^{m_1}}{e^{m_1} - e^{m_2}}, \quad a_3 = \frac{-e^{m_4}}{e^{m_3} - e^{m_4}}, \quad a_4 = \frac{e^{m_3}}{e^{m_3} - e^{m_4}}, \quad a_5 = \frac{-e^{m_6}}{e^{m_5} - e^{m_6}}, \]
\[ a_6 = \frac{e^{m_5}}{e^{m_3} - e^{m_6}}, \quad a_7 = \frac{-S_0\text{ScRe}a_1 m_1^2}{m_1^2 - \text{ScRe}m_1 - \text{ScRe}R}, \quad a_8 = \frac{-S_0\text{ScRe}a_2 m_2^2}{m_2^2 - \text{ScRe}m_2 - \text{ScRe}R}, \]
\[ a_9 = \frac{e^{m_8} + a_7(e^{m_8} - e^{m_1}) + a_8(e^{m_8} - e^{m_2})}{e^{m_5} - e^{m_6}}, \quad a_{10} = \frac{e^{m_3} - a_7(e^{m_3} - e^{m_1}) - a_8(e^{m_3} - e^{m_2})}{e^{m_5} - e^{m_6}}, \]
\[ a_{11} = \frac{-S_0\text{ScRe}a_3 m_3^2}{m_3^2 - \text{ScRe}m_3 - \text{ScRe}(R + i\omega)}, \quad a_{12} = \frac{-S_0\text{ScRe}a_3 m_3^2}{m_3^2 - \text{ScRe}m_3 - \text{ScRe}(R + i\omega)}, \]
\[ a_{13} = \frac{e^{m_10} + a_11(e^{m_10} - e^{m_3}) + a_12(e^{m_10} - e^{m_4})}{e^{m_9} - e^{m_10}}, \quad a_{14} = \frac{e^{m_10} - a_11(e^{m_10} - e^{m_3}) - a_12(e^{m_10} - e^{m_4})}{e^{m_9} - e^{m_10}}, \]
\[ a_{15} = \frac{-S_0\text{ScRe}a_3 m_5^2}{m_5^2 - \text{ScRe}m_5 - \text{ScRe}(R - i\omega)}, \quad a_{16} = \frac{-S_0\text{ScRe}a_6 m_6^2}{m_6^2 - \text{ScRe}m_6 - \text{ScRe}(R - i\omega)}, \]
\[ a_{17} = \frac{e^{m_12} + a_14(e^{m_12} - e^{m_5}) + a_15(e^{m_12} - e^{m_6})}{e^{m_{11}} - e^{m_{12}}}, \quad a_{18} = \frac{e^{m_{11}} - a_14(e^{m_{11}} - e^{m_5}) - a_15(e^{m_{11}} - e^{m_6})}{e^{m_{11}} - e^{m_{12}}}, \]
\[ a_{19} = \frac{-G\text{Re}^2a_1}{m_1^2 - \text{Re}m_1 - (M^2 + \text{Re}/K)}, \quad a_{20} = \frac{-G\text{Re}^2a_2}{m_2^2 - \text{Re}m_2 - (M^2 + \text{Re}/K)}, \]
\[ a_{21} = \frac{-Gm\text{Re}^2a_9}{m_2^2 - \text{Re}m_2 - (M^2 + \text{Re}/K)}, \quad a_{22} = \frac{-Gm\text{Re}^2a_{10}}{m_2^2 - \text{Re}m_2 - (M^2 + \text{Re}/K)}, \]
\[ a_{23} = \frac{-Gm\text{Re}^2a_7}{m_3^2 - \text{Re}m_3 - (M^2 + \text{Re}/K)}, \quad a_{24} = \frac{-Gm\text{Re}^2a_8}{m_3^2 - \text{Re}m_3 - (M^2 + \text{Re}/K)}, \]
\[ a_{25} = a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} + U_0, \]
\[ a_{26} = a_{19} e^{m_1} + a_{20} e^{m_3} + a_{21} e^{m_3} + a_{22} e^{m_5} + a_{23} e^{m_1} + a_{24} e^{m_3} + U_0, \]
\[ a_{27} = \frac{1 - a_{26} + a_{25} e^{m_1}}{e^{m_{14}} - e^{m_{13}}}, \quad a_{28} = \frac{1 - a_{26} + a_{25} e^{m_1}}{e^{m_{14}} - e^{m_{13}}}, \]
\[ a_{29} = \frac{-G\text{Re}^2a_3}{(1 + \alpha\omega\text{Re})m_3^2 - \text{Re}m_3 - (1 + \alpha\omega\text{Re})(M^2 + \text{Re}/K + i\omega\text{Re})}. \]
\begin{align*}
a_{30} &= \frac{-\text{Gr} \text{Re}^2 a_4}{(1 + \alpha i \text{Re})m_4^2 - \text{Re} m_4 - (1 + \alpha i \text{Re}) (M^2 + \text{Re}/K + i \omega \text{Re})}, \\
a_{31} &= \frac{-\text{Gm} \text{Re}^2 a_{13}}{(1 + \alpha i \text{Re})m_5^2 - \text{Re} m_5 - (1 + \alpha i \text{Re}) (M^2 + \text{Re}/K + i \omega \text{Re})}, \\
a_{32} &= \frac{-\text{Gm} \text{Re}^2 a_{14}}{(1 + \alpha i \text{Re})m_{10}^2 - \text{Re} m_{10} - (1 + \alpha i \text{Re}) (M^2 + \text{Re}/K + i \omega \text{Re})}, \\
a_{33} &= \frac{-\text{Gm} \text{Re}^2 a_{11}}{(1 + \alpha i \text{Re})m_3^2 - \text{Re} m_3 - (1 + \alpha i \text{Re}) (M^2 + \text{Re}/K + i \omega \text{Re})}, \\
a_{34} &= \frac{-\text{Gm} \text{Re}^2 a_{12}}{(1 + \alpha i \text{Re})m_4^2 - \text{Re} m_4 - (1 + \alpha i \text{Re}) (M^2 + \text{Re}/K + i \omega \text{Re})}, \\
a_{35} &= a_{29} + a_{30} + a_{31} + a_{32} + a_{33} + a_{34} + U_0, \\
a_{36} &= a_{29} e^{m_1} + a_{30} e^{m_4} + a_{31} e^{m_0} + a_{32} e^{m_0} + a_{33} e^{m_1} + a_{34} e^{m_4} + U_0, \\
a_{37} &= \frac{1}{e^{m_1} - e^{m_0}}, \\
a_{38} &= \frac{1 - a_{36} + a_{35} e^{m_0}}{e^{m_0} - e^{m_1}}, \\
a_{39} &= \frac{-\text{Gr} \text{Re}^2 a_5}{(1 - \alpha i \text{Re})m_2^2 - \text{Re} m_2 - (1 - \alpha i \text{Re}) (M^2 + \text{Re}/K - i \omega \text{Re})}, \\
a_{40} &= \frac{-\text{Gr} \text{Re}^2 a_6}{(1 - \alpha i \text{Re})m_6^2 - \text{Re} m_6 - (1 - \alpha i \text{Re}) (M^2 + \text{Re}/K - i \omega \text{Re})}, \\
a_{41} &= \frac{-\text{Gm} \text{Re}^2 a_{17}}{(1 - \alpha i \text{Re})m_{11}^2 - \text{Re} m_{11} - (1 - \alpha i \text{Re}) (M^2 + \text{Re}/K - i \omega \text{Re})}, \\
a_{42} &= \frac{-\text{Gm} \text{Re}^2 a_{18}}{(1 - \alpha i \text{Re})m_{12}^2 - \text{Re} m_{12} - (1 - \alpha i \text{Re}) (M^2 + \text{Re}/K - i \omega \text{Re})}, \\
a_{43} &= \frac{-\text{Gm} \text{Re}^2 a_{15}}{(1 - \alpha i \text{Re})m_5^2 - \text{Re} m_5 - (1 - \alpha i \text{Re}) (M^2 + \text{Re}/K - i \omega \text{Re})}, \\
a_{44} &= \frac{-\text{Gm} \text{Re}^2 a_{16}}{(1 - \alpha i \text{Re})m_6^2 - \text{Re} m_6 - (1 - \alpha i \text{Re}) (M^2 + \text{Re}/K - i \omega \text{Re})}, \\
a_{45} &= a_{39} + a_{40} + a_{41} + a_{42} + a_{43} + a_{44} + U_0, \\
a_{46} &= a_{39} e^{m_3} + a_{40} e^{m_6} + a_{41} e^{m_1} + a_{42} e^{m_1} + a_{43} e^{m_3} + a_{44} e^{m_6} + U_0, \\
a_{47} &= \frac{1 + a_{46} - a_{45} e^{m_3}}{e^{m_3} - e^{m_1}}, \\
a_{48} &= \frac{1 - a_{46} + a_{45} e^{m_1}}{e^{m_1} - e^{m_3}}.
\end{align*}