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Boundary Layer Theory for a Particulate Suspension

Order of magnitude considerations are employed to develop boundary layer equations for two phase particle/fluid suspension flows. It is demonstrated that a variety of possibilities exist and three of these are examined in detail. Two are applied to the problem of flow past a semi-infinite flat plate.

Introduction

This paper is concerned with boundary layer theory for a particulate suspension. Given the importance of boundary layers in applications, this topic has received surprisingly little attention. There is a history of work on the problem of the steady laminar boundary layer on a semi-infinite flat plate with recent contributions by Osipov (1980), Prabha and Jain (1980), and Wang and Glass (1988). References to earlier work can be found in these papers. In contrast to investigations of the flat plate and a few other specific geometries, there appears to have been little effort devoted to determining the general form of boundary layer equations. The present paper deals with this topic.

A boundary layer order of magnitude analysis is carried out using a typical set of two fluid equations representative of the current literature. It is found that a variety of outcomes are possible depending on the order of magnitude assumptions selected. Three of the most interesting cases are singled out for explicit presentation. Some specific numerical results are then given for the problem of steady laminar boundary layer flow past a semi-infinite flat plate. It is shown that the boundary layer model employed greatly influences predictions.

Governing Equations

The boundary layer analysis to be described in the present work is based on the following typical set of two phase flow equations.

$$\partial_t \phi - \nabla \cdot ((1 - \phi) \mathbf{v}_c) = 0, \quad \partial_t \phi + \nabla \cdot (\phi \mathbf{v}_d) = 0 \quad (1)$$

represent respective balances of mass for the fluid and particulate phases (with the true densities of both phases assumed constant). In Eqs. (1) ∇ is the gradient operator, t is time, ϕ is the particulate volume fraction, \mathbf{v}_c is the fluid phase velocity vector, and \mathbf{v}_d is the particulate phase velocity vector.

$$\rho_c (1 - \phi) (\partial_t \mathbf{v}_c + \mathbf{v}_c \cdot \nabla \mathbf{v}_c) = \nabla \cdot \underline{\underline{\sigma}}_c - \mathbf{f}$$

$$\rho_d \phi (\partial_t \mathbf{v}_d + \mathbf{v}_d \cdot \nabla \mathbf{v}_d) = \nabla \cdot \underline{\underline{\sigma}}_d + \mathbf{f} \quad (2)$$

represent respective balances of linear momentum for the fluid and particulate phases (with external body forces neglected). In Eqs. (2) ρ_c is the fluid true density, ρ_d is the particulate true density, $\underline{\underline{\sigma}}_c$ is the fluid phase stress tensor, $\underline{\underline{\sigma}}_d$ is the particulate phase stress tensor, and \mathbf{f} is the interphase force per unit volume acting on the particulate phase. The balance laws discussed above will be supplemented by the constitutive equations

$$\underline{\underline{\sigma}}_c = -(1 - \lambda \phi) p \underline{\underline{I}} + 2\mu_c (1 - \phi) \underline{\underline{D}}_c; \quad \underline{\underline{D}}_c = (\nabla \mathbf{v}_c + \nabla \mathbf{v}_c^T) / 2$$

$$\underline{\underline{\sigma}}_d = -(\lambda \phi p + q) \underline{\underline{I}} + 2\mu_d \phi \underline{\underline{D}}_d; \quad \underline{\underline{D}}_d = (\nabla \mathbf{v}_d + \nabla \mathbf{v}_d^T) / 2$$

$$\mathbf{f} = \rho_d \phi \mathbf{v}_r / \tau + \lambda p \nabla \phi; \quad \mathbf{v}_r = \mathbf{v}_d - \mathbf{v}_c \quad (3)$$

where p is the indeterminate pressure, q is the particulate phase dynamic pressure, λ is a coefficient which determines the apportionment of the indeterminate pressure gradient between the phases (see below), μ_c and μ_d are dynamic viscosity coefficients, τ is the interphase relaxation time, $\underline{\underline{I}}$ is the unit tensor, and a superposed T indicates the transpose of a second order tensor. In general q , λ , μ_c , μ_d , and τ are functions of such quantities as ϕ , v_r , and the invariants of $\underline{\underline{D}}_c$ and $\underline{\underline{D}}_d$.

The following brief comments about Eqs. (3) are in order. The forms of Eqs. (3b,c) should be such that Eq. (2b) will reduce to the Eulerian form of the equation of motion for a single particle when the volume fraction is sufficiently small. Angular momentum considerations do not require that $\underline{\underline{\sigma}}_c$ and $\underline{\underline{\sigma}}_d$ be individually symmetric but do require that the combination $\underline{\underline{\sigma}}_c + \underline{\underline{\sigma}}_d$ be symmetric (in the absence of body moments). Several particle phase stress mechanisms have been proposed. These include direct contact between particles, wall effects, local particle deformations, and the consequences of the averaging required to model a system containing discrete particles as a continuum. The precise forms of λ , μ_c , μ_d , and q are model dependent. In the present work it is not necessary to restrict attention to a specific model and this has not been done.

The focus of the present paper is on laminar flow. The inclusion of $\underline{\underline{D}}_c$ and $\underline{\underline{D}}_d$ in the list of arguments given above formally allows Eqs. (3) to include algebraic turbulence models. It is highly doubtful, however, that such models are sufficiently inclusive to simulate two phase turbulence under any but the simplest conditions.

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Equations (2) and (3) are representative of several models which have appeared in the literature such as those discussed by Marble (1970), Ungarish and Greenspan (1983), Gidaspow (1986), Ungarish (1988), Foscolo et al. (1989), Tsuo and Gidaspow (1990), Ham et al. (1990), Ganser and Lightbourne (1991), Foscolo et al. (1991), and McCarthy (1991). These equations are, however, by no means meant to be all inclusive. In fact, several important phenomena (lift, added mass, and gravity, for example) have been purposely omitted in order to create a model containing only the most fundamental two phase flow effects. It is felt that a good understanding of this case is essential to establish a baseline for future studies of more complicated models. The omission of a given physical effect should not be interpreted as a statement that this effect is always negligible in boundary layer situations.

The primary purpose of the present work is to illustrate the variety of boundary layer equations that can be produced by different order of magnitude assumptions. For this reason, attention will be confined herein to a relatively simple case. It will be assumed that the boundary layer exists on a flat wall and can be characterized by free stream velocity v_∞ , pressure p_∞ , and volume fraction ϕ_∞ and a characteristic length L . It will further be assumed that q , μ_c , μ_d , and τ are functions only of ϕ and that λ is constant (either zero or unity). Finally, attention will be confined to steady flow. Extension to more complicated cases is straightforward.

Let x and y be respective coordinates tangent and normal to the surface and $v_c = \mu_c/\rho_c$ and $v_d = \mu_d/\rho_d$ be kinematic viscosities. The Reynolds number is defined to be

$$N_1 = Lv_\infty/v_c(\phi_\infty) = 1/\epsilon^2 \quad (4)$$

Let

$$\begin{aligned} x &= Ls, \quad y = \epsilon Ln, \quad p = \rho_c v_\infty^2 H(s, n), \quad q = \rho_c v_\infty^2 J(\phi), \quad \phi = \phi_\infty Q(s, n) \\ v_c &= v_\infty(\mathbf{e}_x U_c(s, n) + \mathbf{e}_y \epsilon V_c(s, n)) \\ v_d &= v_\infty(\mathbf{e}_x U_d(s, n) + \mathbf{e}_y \epsilon V_d(s, n)) \end{aligned} \quad (5)$$

with \mathbf{e} denoting a unit vector. Substituting Eqs. (5) into Eqs. (1-3), combining the results, and rearranging leads to the equations

$$\begin{aligned} \partial_s((1 - \phi_\infty Q)U_c) + \partial_n((1 - \phi_\infty Q)V_c) &= 0 \\ (1 - \phi_\infty Q)(U_c \partial_s U_c + V_c \partial_n U_c) + (1 - \lambda \phi_\infty Q) \partial_s H \\ + N_2 \phi_\infty Q C_2 (U_c - U_d) - (\epsilon^2 \partial_s((1 - \phi_\infty Q)C_1(2\partial_s U_c + \partial_n V_c)) \\ + \partial_n((1 - \phi_\infty Q)C_1 \partial_n U_c)) &= 0 \\ \epsilon^2(1 - \phi_\infty Q)(U_c \partial_s V_c + V_c \partial_n V_c) + (1 - \lambda \phi_\infty Q) \partial_n H \\ + \epsilon^2 N_2 \phi_\infty Q C_2 (V_c - V_d) - \epsilon^2(\epsilon^2 \partial_s((1 - \phi_\infty Q)C_1 \partial_s V_c) \\ + \partial_n((1 - \phi_\infty Q)C_1(\partial_s U_c + 2\partial_n V_c))) &= 0 \end{aligned} \quad (6)$$

for the fluid phase and

$$\begin{aligned} \partial_s(QU_d) + \partial_n(QV_d) &= 0 \\ Q(U_d \partial_s U_d + V_d \partial_n U_d) + (\lambda Q \partial_s H + C_3 \partial_s Q)/N_2 \\ + QC_2(U_d - U_c) - (\epsilon^2 \partial_s(QC_2 2\partial_s U_d + \partial_n V_d)) \\ + \partial_n(QC_4 \partial_n U_d) &= 0 \\ \epsilon^2 Q(U_d \partial_s V_d + V_d \partial_n V_d) + (\lambda Q \partial_n H + C_3 \partial_n Q)/N_2 \\ + \epsilon^2 QC_2(V_d - V_c) - \epsilon^2(\epsilon^2 \partial_s(QC_4 \partial_s V_d) \\ + \partial_n(QC_4(\partial_s U_d + 2\partial_n V_d))) &= 0 \end{aligned} \quad (7)$$

for the particle phase. In Eqs. (6) and (7)

$$\begin{aligned} C_1(Q) &= v_c(\phi_\infty Q)/v_c(\phi_\infty), \quad C_2(Q) = L/(v_\infty \tau(\phi_\infty Q)) \\ C_3(Q) &= J'(\phi_\infty Q), \quad C_4(Q) = v_d(\phi_\infty Q)/v_c(\phi_\infty) \end{aligned} \quad (8)$$

are functions of the normalized volume fraction Q and

$$N_2 = \rho_d/\rho_c \quad (9)$$

is the true density ratio. A prime denotes the derivative of a function of one variable with respect to its argument. It can be seen from Eqs. (6) and (7) that if $\lambda = 0$ the entire indeterminate pressure gradient is assigned to the fluid phase while if $\lambda = 1$ it is shared between the two phases in proportion to their volume fractions. Both of these formulations have appeared in the literature. Equations (6) and (7) form the basis for the order of magnitude analysis to be carried out in the next section.

Most of the dimensionless quantities appearing in this paper do not have well established names at present. A consistent notation has, therefore, been adopted in which all dimensionless functions are denoted by C 's and all dimensionless numbers are denoted by N 's. It is hoped that the reader will not find this notation confusing. The function C_1 is a measure of the variation of fluid phase viscosity with volume fraction. The function C_2 is a measure of the variation of relaxation effects with volume fraction. It can be viewed as the ratio of the characteristic time L/v_∞ to the relaxation time of a homogeneous particle phase released with speed v_∞ in a fluid held at rest. Microscopic models indicate that C_2 is proportional to the ratio of L to a characteristic particle dimension. While this ratio is large, the coefficient of proportionality can realistically take on any magnitude (depending on flow geometry and flow conditions). Thus, C_2 is not necessarily large. The function C_3 plays the role of a bulk modulus of compressibility for the particle phase. The function C_4 is a measure of the variation of particle phase viscosity with volume fraction. In some models (see, for instance, Soo (1967)) an identification is made between the particle phase kinematic viscosity and the particle phase diffusion coefficient (see below). If this is done, C_4 becomes the local inverse Schmidt number.

Order of Magnitude Analysis

Boundary layer equations are found by taking the limits of Eqs. (6) and (7) as $\epsilon \rightarrow 0$. This will be done herein assuming that $C_1 = O(1)$ and $C_4 = O(1)$ (thus omitting, for the sake of concreteness, many interesting possibilities at the outset) but leaving the magnitudes of C_2 and C_3 arbitrary. Then the mass balances (6a) and (7a) remain

$$\begin{aligned} \partial_s((1 - \phi_\infty Q)U_c) + \partial_n((1 - \phi_\infty Q)V_c) &= 0 \\ \partial_s(QU_d) + \partial_n(QV_d) &= 0; \end{aligned} \quad (10)$$

Eqs. (6b) and (7b) yield the tangential linear momentum balances

$$\begin{aligned} (1 - \phi_\infty Q)(U_c \partial_s U_c + V_c \partial_n U_c) \\ + (1 - \lambda \phi_\infty Q) \partial_s H + N_2 \phi_\infty Q C_2 (U_c - U_d) \\ - \partial_n((1 - \phi_\infty Q)C_1 \partial_n U_c) = 0 \\ Q(U_d \partial_s U_d + V_d \partial_n U_d) + (\lambda Q \partial_s H + C_3 \partial_s Q)/N_2 + QC_2(U_d - U_c) \\ - \partial_n(QC_4 \partial_n U_d) = 0; \end{aligned} \quad (11)$$

and Eqs. (6c) and (7c) yield the normal linear momentum balances

$$\begin{aligned} \partial_n H + \epsilon^2 N_2 \phi_\infty Q C_2 (V_c - V_d)/(1 - \lambda \phi_\infty Q) = 0 \\ C_3 \partial_n Q + \epsilon^2 N_2 Q (U_d \partial_s V_d + V_d \partial_n V_d + C_2 (V_d - V_c)/(1 - \lambda \phi_\infty Q)) \\ - \partial_n(QC_4(\partial_s U_d + 2\partial_n V_d)) = 0 \end{aligned} \quad (12)$$

The terms multiplied by ϵ^2 in Eqs. (12) have been retained to allow for a variety of orders of magnitude of C_2 and C_3 . Three interesting cases will be discussed below.

The first situation to be considered is

$$C_2 = O(1), \quad C_3 = O(1) \quad (13)$$

Then taking the limit of Eqs. (12) as $\epsilon \rightarrow 0$ and solving the resulting equations yields

$$H = H_\infty(s), \quad Q = Q_\infty(s) \quad (14)$$

where H_∞ and Q_∞ are, respectively, the values of H and Q at the edge of the boundary layer (as determined from an inviscid analysis). Now Eqs. (10) and the results of substituting Eqs. (14) into Eqs. (11) constitute four equations in which H , Q , C_1 , C_2 , C_3 , and C_4 are known functions of s and the dependent variables are U_c , V_c , U_d , and V_d . The equations for each phase closely resemble the boundary layer equations for single phase flow. That is the primary reason for selecting order of magnitude assumptions (13) for attention.

A second situation of interest is that of

$$C_2 = O(1), C_3 = O(\epsilon^2) \quad (15)$$

Then the limit of Eq. (12a) can be taken as $\epsilon \rightarrow 0$ to yield Eq. (14a). This, in turn, renders determinate the terms in Eqs. (11) involving H . For convenience, let

$$C_3 = \epsilon^2 C_5 / N_2; C_5 = C_5(Q) = O(1) \quad (16)$$

Substituting Eq. (16) into Eqs. (11b) and (12b), dividing the latter by ϵ^2 , and taking the limits as $\epsilon \rightarrow 0$ produces, respectively,

$$\begin{aligned} Q(U_d \partial_s U_d + V_d \partial_n U_d) + Q C_2 (U_d - U_c) - \partial_n (Q C_4 \partial_n U_d) = 0 \\ Q(U_d \partial_s V_d + V_d \partial_n V_d) + C_5 \partial_n Q + Q C_2 (V_d - V_c) / (1 - \lambda \phi_\infty Q) \\ - \partial_n (Q C_4 (\partial_s U_d + 2 \partial_n V_d)) = 0 \end{aligned} \quad (17)$$

Now Eqs. (10), (11a), and (17) are a set of five equations in which H is a known function of s and the dependent variables are U_c , V_c , U_d , V_d , and Q . It should be pointed out that the particle phase normal momentum balance appears as one of the equations to be solved. This phenomenon (somewhat unusual in boundary layer formulations) is also a feature of the dusty gas model discussed by Marble (1970) (and often employed when the volume fraction of particulate material is expected to be small). The boundary layer theory associated with order of magnitude assumptions (15) can, therefore, be thought of as a generalization of dusty gas boundary layer theory to account for finite volume fractions. This is the primary reason for interest in these order of magnitude assumptions.

The third situation to be discussed herein is that of

$$C_2 = O(1/\epsilon^2), C_3 = O(1) \quad (18)$$

To avoid obtaining a singular limit under these circumstances one must add Eq. (11a) to $\phi_\infty N_2 \times$ Eq. (11b), add Eq. (12a) to $\phi_\infty N_2 \times$ Eq. (12b), then take the limit of the results as $\epsilon \rightarrow 0$ to get, respectively,

$$\begin{aligned} (1 - \phi_\infty Q)(U_c \partial_s U_c + V_c \partial_n U_c) + \phi_\infty N_2 Q(U_d \partial_s U_d \\ + V_d \partial_n U_d) + \partial_s (H + J) - \partial_n ((1 - \phi_\infty Q) C_1 \partial_n U_c \\ + \phi_\infty N_2 Q C_4 \partial_n U_d) = 0 \\ \partial_n (H + J) = 0 \end{aligned} \quad (19)$$

where Eq. (8c) has been used. Equation (19b) can be solved to yield

$$H + J = H_\infty(s) + J(\phi_\infty Q_\infty(s)) \quad (20)$$

Making the substitution

$$C_2 = (1 - \lambda \phi_\infty Q) C_3 / (\epsilon^2 N_2 C_6); C_6 = C_6(Q) = O(1) \quad (21)$$

in Eqs. (11b) and (12b) and taking the limits of the results as $\epsilon \rightarrow 0$ yields, respectively,

$$U_d = U_c, V_d = V_c - C_6 \partial_n (\ln Q) \quad (22)$$

Substituting Eqs. (22) into Eq. (10b) and the combination of Eqs. (19a) and (20) produces the respective results

$$\begin{aligned} \partial_s (Q U_c) + \partial_n (Q V_c) - \partial_n (C_6 \partial_n Q) = 0 \\ (1 - \phi_\infty Q + N_2 \phi_\infty Q)(U_c \partial_s U_c + V_c \partial_n U_c) \\ + H_\infty'(s) + \phi_\infty C_3 (\phi_\infty \partial_\infty(s)) Q_\infty'(s) - (\partial_n ((1 - \phi_\infty Q) C_1 \\ + \phi_\infty N_2 Q C_4) \partial_n U_c) + \phi_\infty N_2 C_6 \partial_n Q \partial_n U_c = 0 \end{aligned} \quad (23)$$

Equations (10a) and (23) are a set of three equations with dependent variables U_c , V_c , and Q . Once these equations have

been solved, U_d and V_d can be determined from Eqs. (22).

The boundary layer equations consistent with order of magnitude assumptions (18) represent a generalization of the usual convection/diffusion model of particle transport to account for a finite volume fraction. It is this feature that makes order of magnitude assumptions (18) of interest. Equations (22) are equivalent to the boundary layer forms of Fick's law of diffusion with a dimensionless diffusion coefficient C_6 . In particular, if ν_c becomes constant as its argument becomes small, assuming $\phi_\infty \ll 1$ makes $C_1 = 1$ (from Eq. (8a)). Then Eqs. (10a) and (23) reduce to

$$\begin{aligned} \partial_s U_c + \partial_n V_c = 0, U_c \partial_s U_c + V_c \partial_n U_c + H_\infty'(s) - \partial_{nn} U_c = 0 \\ U_c \partial_s Q + V_c \partial_n Q - \partial_n (C_6 \partial_n Q) = 0 \end{aligned} \quad (24)$$

where Eq. (24a) has been used in Eq. (23a) to obtain Eq. (24c). Equations (24a,b) are the usual single phase boundary layer equations and Eq. (24c) is the usual convection/diffusion equation for a passive scalar.

It should be mentioned that the three special cases discussed above by no means exhaust all physically plausible possibilities. They were chosen both to illustrate the variety of possible formulations and because each has interesting features. It should also be recognized that the magnitude of some or all of the C 's may vary significantly due to relatively modest changes in Q (see, for example, Gidaspo (1986) and Foscolo et al. (1991)). Under these circumstances there may be no single set of order of magnitude assumptions which is appropriate throughout the entire flow field. Then extreme care would be necessary in applying the concepts of boundary layer theory.

Flat Plate Boundary Layer

To illustrate the application of some of the equations developed in the previous section, attention will be directed to the flow past a semi-infinite flat plate subjected to zero pressure gradient with its leading edge at $(s, n) = (0, 0)$. This problem is of both theoretical and practical interest in itself and, in addition, serves as a first approximation for many flow situations involving ducts and wings. In this case the inviscid flow is simply a uniform stream characterized by

$$U_{c,\infty} = U_{d,\infty} = 1, V_{c,\infty} = v_{d,\infty} = 0, H_\infty = 1, Q_\infty = 1 \quad (25)$$

Since the convection/diffusion problem associated with flow past a flat plate is well known, attention will be confined herein to the first two boundary layer formulations discussed in the previous section.

First, consider a solution based on order of magnitude assumptions (13). Equations (14) will now read

$$H = Q = 1 \quad (26)$$

Substituting Eq. (26b) into Eqs. (8) yields

$$C_1 = 1, C_2 = 1, C_3 = 0, C_4 = \nu_d / \nu_c = N_3 \quad (27)$$

where the choice $L = \nu_\infty \tau$ has been used to get Eq. (27b) (since there is no natural characteristic length associated with flow past a semi-infinite flat plate). Equation (27b) illustrates the point made earlier that C_2 need not be large.

Substituting Eqs. (26) and (27) and the modified Blasius transformations

$$s = \xi / (1 - \xi), n = (2\xi / (1 - \xi))^{1/2} \eta$$

$$U_c = F_c(\xi, \eta), V_c = ((1 - \xi) / (2\xi))^{1/2} (G_c(\xi, \eta) + \eta F_c(\xi, \eta))$$

$$U_d = F_d(\xi, \eta), V_d = ((1 - \xi) / (2\xi))^{1/2} (G_d(\xi, \eta) + \eta F_d(\xi, \eta)) \quad (28)$$

into Eqs. (10) and (11) yields

$$\partial_\eta G_c + F_c + 2\xi(1 - \xi) \partial_\xi F_c = 0$$

$$\partial_\eta G_d + F_d + 2\xi(1 - \xi) \partial_\xi F_d = 0$$

$$\partial_{\eta\eta} F_c - G_c \partial_\eta F_c - 2\xi(1 - \xi) F_c \partial_\xi F_c + 2N_4 \xi (F_d - F_c) / (1 - \xi) = 0$$

$$N_3 \partial_{\eta\eta} F_d - G_d \partial_\eta F_d - 2\xi(1 - \xi) F_d \partial_\xi F_d$$

$$+ 2\xi(F_c - F_d) / (1 - \xi) = 0 \quad (29)$$

where

$$N_4 = N_1 \phi_\infty / (1 - \phi_\infty) = \rho_d \phi_\infty / (\rho_c (1 - \phi_\infty)) \quad (30)$$

is the particle loading. The boundary and matching conditions employed to solve Eqs. (29) were

$$\begin{aligned} F_c(\xi, 0) = 0, \quad G_c(\xi, 0) = 0 \\ F_d(\xi, 0) = N_5((1 - \xi)/(2\xi))^{1/2} \partial_\eta F_d(\xi, 0), \quad G_d(\xi, 0) = 0 \\ F_c(\xi, \eta) \rightarrow 1, \quad F_d(\xi, \eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty \end{aligned} \quad (31)$$

where N_5 is a particle phase wall slip parameter. Equation (31c) allows for particle phase wall slip in a manner similar to that used in rarefied gas dynamics. In reality the particle phase wall slip velocity is controlled by a variety of physical effects such as sliding friction, rolling friction, the nature of particle/surface collisions, particle shapes, etc. It is not possible to model such effects with precision at present, but by adjusting the slip parameter N_5 it is at least possible to produce a wide variety of wall slip profiles. Equations (31c,d) should be dropped if particle phase viscosity is omitted from the model.

For the subsequent presentation of numerical results it is convenient to define the respective skin friction coefficients of the fluid and particulate phases as

$$C_c(\xi) = \partial_\eta F_c(\xi, 0), \quad C_d(\xi) = N_3 \partial_\eta F_d(\xi, 0) \quad (32)$$

and the respective displacement thickness coefficients of the fluid and particulate phases as

$$\Delta_c(\xi) = \int_0^\infty (1 - F_c(\xi, \eta)) d\eta, \quad \Delta_d(\xi) = \int_0^\infty (1 - F_d(\xi, \eta)) d\eta \quad (33)$$

These parameters were selected (from many available) to illustrate parametric trends.

Numerical solutions of Eqs. (29) subject to Eqs. (31) were computed using an extension of the methodology described by Blottner (1970) to two phase flow. Some typical results are presented in Figs. 1–6. For these calculations N_5 was taken to be zero, thus eliminating particle phase wall slip.

Figures 1 and 2 show representative tangential velocity profiles. They illustrate the transition from frozen to equilibrium behavior which is a characteristic of two phase flows. Near $\xi = 0$ the effect of interphase drag is negligible and each phase moves independently of the other (frozen flow). Near $\xi = 1$ the effect of interphase drag dominates and both phases move with the same speed (equilibrium flow). This type of transition is not confined to boundary layer flows but appears, in one form or another, in a variety of two phase flow situations.

Figures 3–6 present typical distributions of the displacement thickness and skin friction coefficients. To allow a transition from fluid/fluid behavior at small volume fractions to fluid/solid behavior at volume fractions close to maximum packing requires a particle phase viscosity function which increases rapidly with volume fraction. According to this interpretation, the results corresponding to the larger values of N_3 can be thought of as representing finite volume fraction situations.

It was possible to obtain numerical solutions to Eqs. (29) without difficulty for all parametric combinations attempted. The results presented in Figs. 1–6 are representative of these computations.

Second, consider a solution based on the order of magnitude assumptions (15) and the further assumptions $\phi \ll 1$, $\rho_d/\rho_c \gg 1$, and $\lambda = 0$ (inherent in the dusty gas model discussed by Marble (1970)). Then making the normal assumptions that $\nu_c(\phi)$, $\nu_d(\phi)$, $\tau(\phi)$ and $J'(\phi)$ approach constants as their arguments approach zero (in Eqs. (8) and (16)) leads to Eqs. (27a,b,d) and

$$C_5 = J'/\epsilon^2 = N_6 \quad (34)$$

Using the assumptions mentioned at the beginning of this paragraph and substituting Eqs. (26a), (27a,b,d), (28), and (34) into Eqs. (10), (11a), and (17) yields

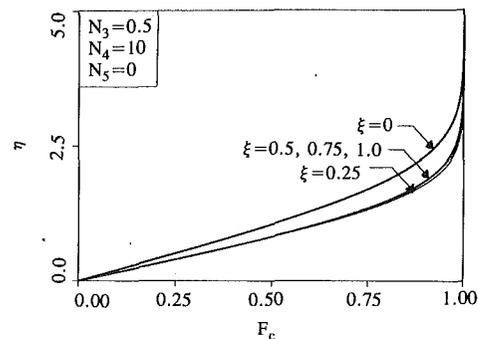


Fig. 1 Fluid phase tangential velocity profiles

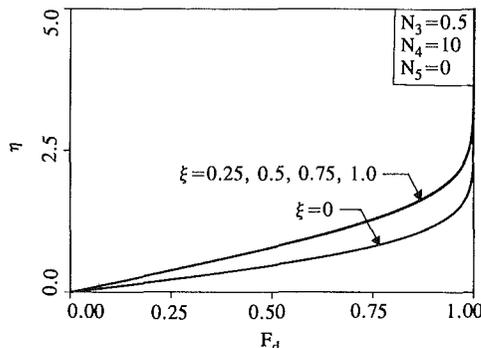


Fig. 2 Particle phase tangential velocity profiles

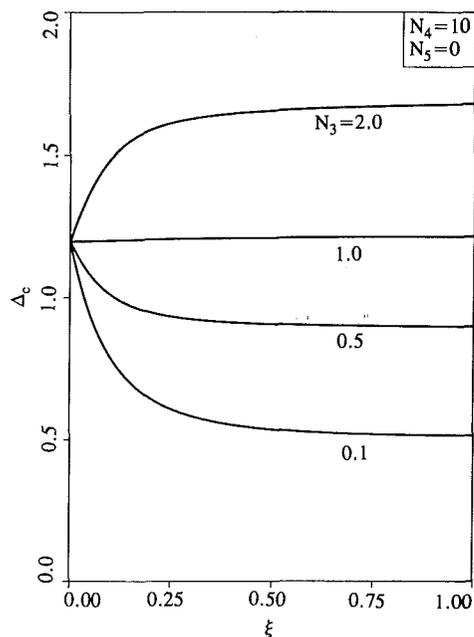


Fig. 3 Fluid phase displacement thickness coefficient versus position

$$\begin{aligned} \partial_\eta G_c + F_c + 2\xi(1 - \xi)\partial_\xi F_c = 0 \\ (\partial_\eta G_d + F_d + 2\xi(1 - \xi)\partial_\xi F_d)Q + G_d \partial_\eta Q + 2\xi(1 - \xi)F_d \partial_\xi Q = 0 \\ \partial_{\eta\eta} F_c - G_c \partial_\eta F_c - 2\xi(1 - \xi)F_c \partial_\xi F_c + 2N_4 \xi Q(F_d - F_c)/(1 - \xi) = 0 \\ N_3(\partial_{\eta\eta} F_d + \partial_\eta(\ln Q)\partial_\eta F_d) - G_d \partial_\eta F_d - 2\xi(1 - \xi)F_d \partial_\xi F_d \\ + 2\xi(F_c - F_d)/(1 - \xi) = 0 \\ N_3(2(\partial_{\eta\eta} G_d + \partial_\eta(\ln Q)\partial_\eta G_d) + 3\partial_\eta F_d + \partial_\eta(\ln Q)F_d \\ + 2\xi(1 - \xi)(\partial_{\xi\eta} F_d + \partial_\xi(\ln Q)\partial_\eta F_d) - G_d \partial_\eta G_d - 2\xi(1 - \xi)F_d \partial_\xi G_d \\ + \eta F_d^2 + 2\xi(G_c - G_d - N_6 \partial_\eta(\ln Q))/(1 - \xi) = 0 \end{aligned} \quad (35)$$

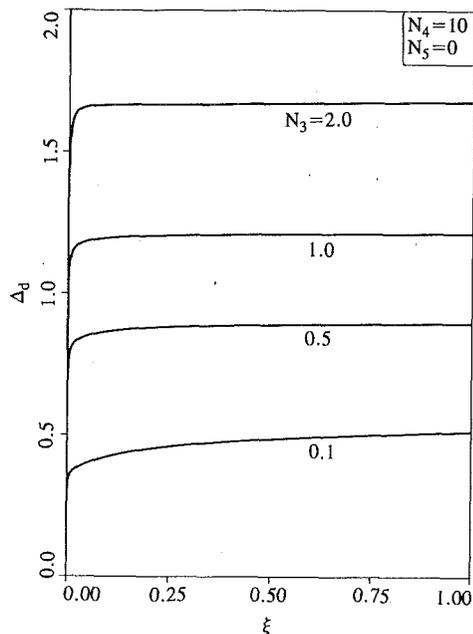


Fig. 4 Particle phase displacement thickness coefficient versus position

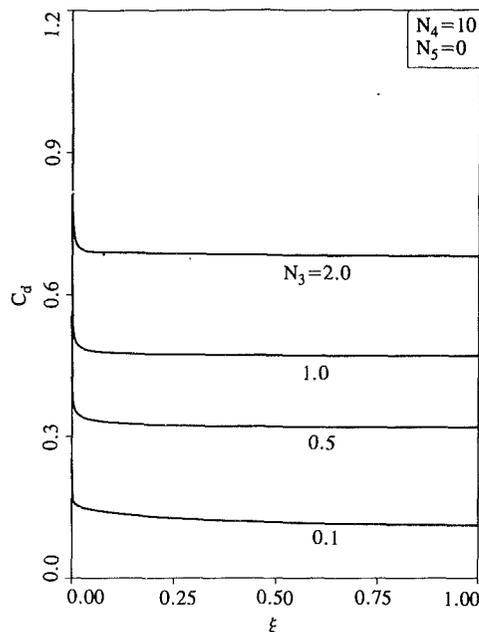


Fig. 6 Particle phase skin friction coefficient versus position

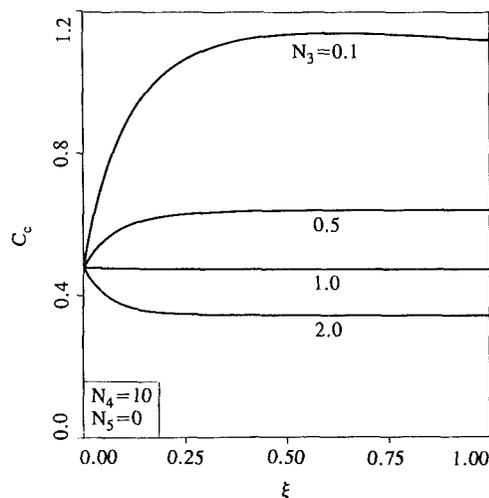


Fig. 5 Fluid phase skin friction coefficient versus position

where Eq. (30) for the particle loading now assumes the small volume fraction form $N_4 = \phi_\infty N_1 = 0(1)$. Equations (35) are to be solved subject to boundary and matching conditions (31) supplemented by

$$G_d(\xi, \eta) \rightarrow G_c(\xi, \eta), \quad Q(\xi, \eta) \rightarrow 1 \quad \text{as } \eta \rightarrow \infty \quad (36)$$

The dusty gas equations follow from Eqs. (35) by equating N_3 and N_6 to zero. Thus, Eqs. (35) generalize the dusty gas equations to allow for particulate phase stresses while retaining the small volume fraction assumption.

In contrast to the well-behaved nature of Eqs. (29), no combination of the parameters N_3 , N_4 , N_5 , and N_6 was found for which a numerical solution to Eqs. (35) exhibited a positive bounded Q throughout the flow field. This phenomenon has already been observed by Osipov (1980), Prabha and Jain (1980), and Wang and Glass (1988) in their work using the dusty gas equations ($N_3 = N_6 = 0$) where Q always became infinite in the vicinity of $\xi = 0.5$. For all solutions attempted in the present work it was found that Q would either become extremely large or become negative somewhere in the range $0 < \xi < 1$. Many parametric combinations were tried without success.

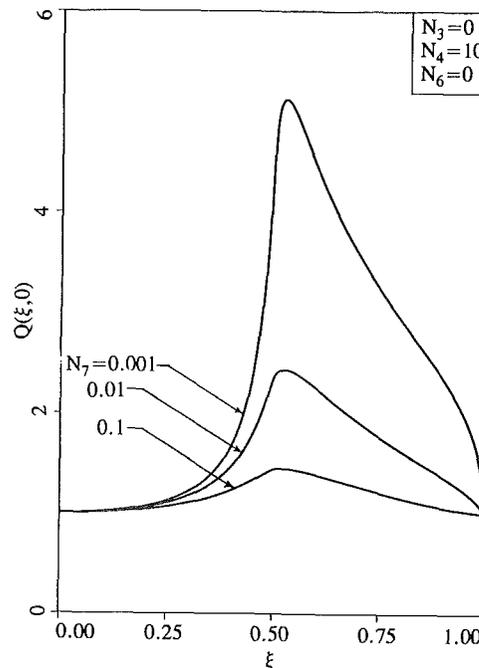


Fig. 7 Particle phase normalized wall volume fraction versus position

The addition of a fictitious diffusion term $N_7 \partial_{\eta\eta} Q$ to the right-hand side of Eq. (35b) (and the imposition of an associated boundary condition $\partial_\eta Q(\xi, 0) = 0$) was found to control the behavior of Q (as previously reported in a different context by Ungarish and Greenspan, 1983). This device was employed to obtain numerical solutions which could be used to illustrate the behavior observed. Some typical results are reported in Figs. 7-10.

Figures 7 and 8 present computations based on the dusty gas model. Figure 7 shows the approach of the wall volume fraction to singular behavior in the vicinity of $\xi = 0.5$ as the fictitious diffusion coefficient N_7 is reduced. For $N_7 = 0$ a continuous solution could not be found. Figure 8 exhibited corresponding values of the fluid phase skin friction coefficient. It can be seen that this quantity is only weakly influenced

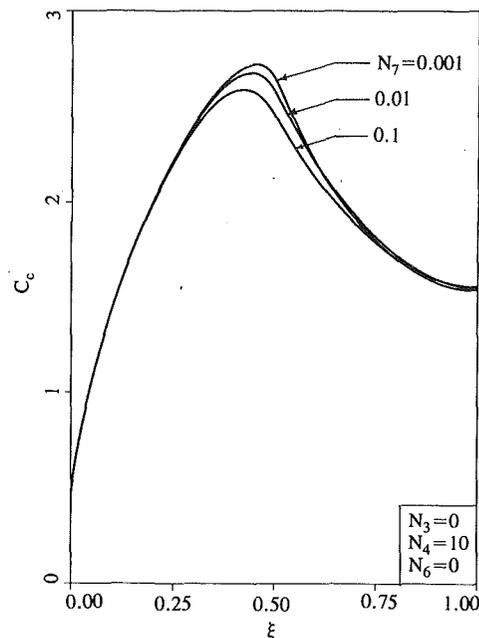


Fig. 8 Fluid phase skin friction coefficient versus position

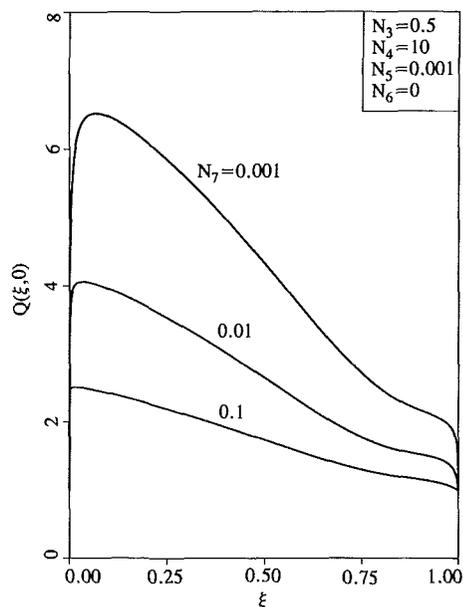


Fig. 9 Particle phase normalized wall volume fraction versus position

by the value of N_7 . This is typical of the behavior of all quantities other than Q .

Figures 9 and 10 show results which illustrate the influence of particle phase stresses. For these calculations a small value of N_5 was used, thus eliminating particle phase wall slip except in the immediate vicinity of the plate's leading edge. Figure 9 indicates the formation of a singularity in the wall volume fraction near the leading edge as N_7 reduced. For $N_7 = 0$ no solution was found. As in the situation discussed in the previous paragraph, the volume fraction was the only quantity found to be strongly affected by the value of N_7 . This is illustrated by Fig. 10 in which the fluid phase skin friction coefficient is chosen as representative.

As mentioned previously, many important physical effects have been omitted from the model employed herein. Some of these may have an effect on the singular behavior reported above. This matter deserves to be pursued but was felt to be beyond the scope of the present work which was to discuss the

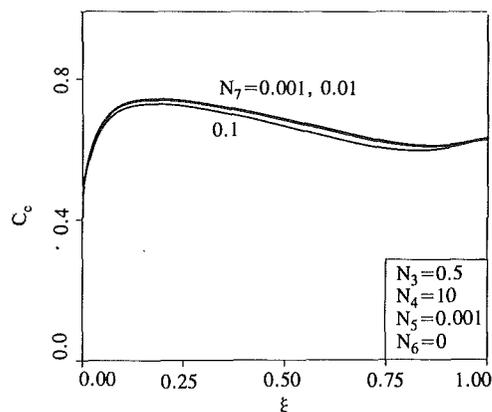


Fig. 10 Fluid phase skin friction coefficient versus position

structure of two phase boundary layer equations rather than give exhaustive results for a specific problem. (It should be mentioned that a few preliminary calculations indicated that the inclusion lift forces did not affect the existence of singularities.)

The singular behavior observed is probably indicative of the formation of a packed bed of particles (see, for instance, Soo, 1967) or a particle free zone (see, for instance, Young and Hanratty, 1991) near the plate surface. To predict either of these phenomena would require a model capable of dealing with the entire range of volume fractions. Since Eqs. (35) are based on the assumption that the volume fraction is small, it would appear better to base boundary layer calculations on a set of equations which allows for a finite volume fraction and to investigate effects omitted from the present model in that context. All that can be said with certainty at this point is that Eqs. (35) do not appear to admit self consistent solutions for the flat plate problem.

In the present work the term multiplied by N_7 was treated as a purely mathematical device employed to achieve the computational goal discussed above. It is referred to, therefore, as a fictitious diffusion term. It is interesting, however, to speculate on potential physical interpretations of this term. Some discussion of this matter is contained in the next two paragraphs.

It is possible that Figs. 7 and 9 are indicative of the strongly discontinuous behavior which can be exhibited by a medium devoid of pressure (see, for example, Kraiko, 1979, 1982). If so, fictitious diffusion could be interpreted as a device to smooth these discontinuities. This issue, while beyond the scope of the present work, is an interesting one which deserves to be pursued.

The effect of fictitious diffusion is of interest for two additional reasons. First, many numerical methods employ artificial diffusion either directly or indirectly (by upwind differencing, for example). It is possible, therefore, that such numerical methods could produce a bounded continuous solution to the problem of steady flow past a flat plate when, in fact, none should exist. Second, many two phase turbulence models (see, for example, Pourahmadi and Humphrey, 1983; Elghobashi et al., 1984; Chen and Wood, 1986; and Rizk and Elghobashi, 1989) contain diffusion terms in the particle phase mass balance. It is possible that the presence of such terms is critical to the existence of solutions in these models.

The results presented in this section show that physically plausible changes in fluid/particle suspension models can lead to significant qualitative (not just quantitative) changes in predictions. This makes clear the need for pertinent experimental data which can be used for model verification. The present authors were unable to locate any experimental work in the literature dealing with laminar flow of a particulate suspension past a flat plate. Such information, when it becomes available,

will be quite useful in deciding the most fruitful directions for extension of the work reported herein.

Conclusion

In this paper the issue of developing boundary layer equations for fluid/particle two phase flows was addressed. It was shown that a variety of possibilities exist. This is, of course, a manifestation of the fact that the number of dimensionless parameters required to characterize the behavior of a two phase system is considerably greater than that required to characterize a single phase system. Three specific examples of boundary layer equations were given and related to previous work. Numerical solutions to the problem of flow past a semi-infinite flat plate subjected to zero pressure gradient were given based on two of the boundary layer formulations developed. It was shown that the predictions associated with the two formulations were quite different.

The present work was confined to plane steady flow past a flat surface. It is believed, however, that the basic findings reported herein are also relevant to flows involving such phenomena as unsteadiness, three dimensionality, and surface curvature. In addition, the results of this investigation should be directly applicable to other thin layer flows such as those occurring in ducts, jets, and wakes.

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