International Journal of Fluid Mechanics Research, Vol. 31, No. 1, 2004

Oscillatory Flow and Heat Transfer in Two Immiscible Fluids†

Ali J. Chamkha
Manufacturing Engineering Department,
The Public Authority for Applied Education and Training
Shuweikh, 70654, Kuwait

J. C. Umavathi and **Abdul Mateen**
Department of Mathematics Gulbarga University
Gulbarga, 585106, Karnataka, India

The problem of unsteady laminar flow and heat transfer of two viscous immiscible fluids through a horizontal channel with permeable walls is investigated for the case of time-dependent oscillatory transpiration velocity. The partial differential equations governing the flow and heat transfer are solved analytically using two-term harmonic and non-harmonic functions in both regions of the channel. Effects of physical parameters such as viscosity ratio, conductivity ratio and Prandtl number on the velocity and temperature fields are depicted graphically while the influence of oscillation amplitude and frequency on the flow velocity and temperature is shown in a tabular form.

* * *

Introduction

The flow and heat transfer problems of viscous fluids through a permeable-walled passages is important in many spheres of science and engineering. The permeable-walled ducts are used in heat exchangers, solar energy collectors, transpiration cooling of gas-turbine blades, combustion chambers, exhaust nozzles, porous-walled flow reactors [2]. Also, porous or permeable walls were used in the past to simulate a variety of surface mechanisms. These include natural transpiration, phase sublimation, propellant burning, ablation cooling, and uniformly distributed irrigation. Such mechanisms take place in a number of interesting models of bio-circulatory systems, flow filtration, chemical dispensing, rocket propellant combustion, and other membrane separation processes [3]. Investigations of laminar porous channel flows were initiated by Berman in 1953 [1]. After Berman's work several studies on the topic followed. For example, Zaturska et al. reported on the flow of a viscous fluid driven along a channel by suction at porous walls [4]. A two-dimensional flow of a viscous fluid in a channel with porous walls was considered in [5]. More recently, King performed an asymptotic analysis of the steady-state and time-dependent Berman problem [6].

†Received 09.01.2004

ISSN 1064-2277

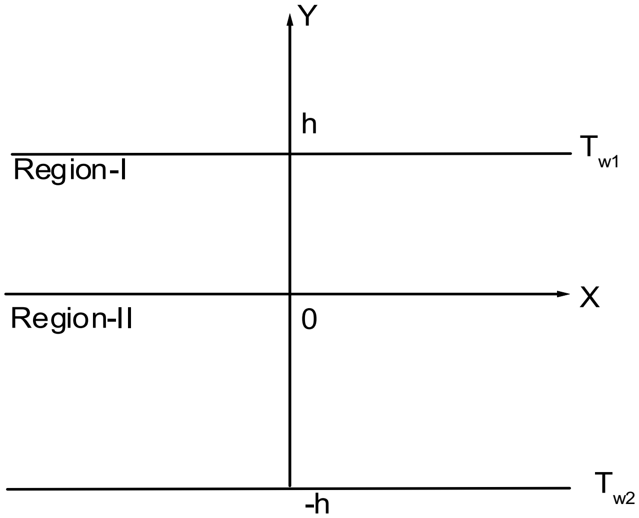

Fig. 1. Physical configuration.

Problems involving immiscible multi-phase flow, heat transfer and multi-component mass transfer also arise in a number of scientific and engineering disciplines. Important applications include petroleum industry, geophysics and plasma physics. In modeling such problems, the presence of a second immiscible fluid phase adds a number of complexities as to the nature of interacting transport phenomena and interface conditions between phases. In general, multi-phase flows are driven by gravitational and viscous forces. There were some theoretical and experimental works on stratified laminar flow of two immiscible fluids in a horizontal pipe [7–9]. A two-phase MHD flow and heat transfer in a parallel plate channel with one of the fluids being electrically conducting was studied in [10]. Two-phase MHD flow and heat transfer in an inclined channel was investigated in [11]. Chamkha reported analytical solutions for the flow of two-immiscible fluids in porous and non-porous parallel-plates channels [12]. Later on, magnetohydrodynamic two-fluid convective flow and heat transfer in composite porous medium was analyzed in [13–15].

All of the above mentioned studies pertain to a steady flow. However, a significant number of problems of practical interest are unsteady. The flow unsteadiness may be caused by a change either in the free stream velocity, or in the surface temperature (surface heat flux), or in both. When there is an impulsive change in the velocity field, the inviscid flow is developed instantly, but the flow in the viscous layer near the wall is developed slowly which becomes a fully developed steady flow after sometime. The effects of free stream oscillations and free convection currents on the flow past an infinite vertical plate with constant suction were studied in [16, 17]. Soundalgekar studied free connvection effects on oscillatory flow past a vertical plate with suction [17]. The oscillatory MHD channel flow and a heat transfer under a transverse magnetic field were analyzed in [18]. The unsteady flows in a semiinfinite expanding pipe with injection through a wall were considered in [19]. An oscillatory flow of unsteady convective fluid in an infinite vertical porous stratum was studied in [20]. Recently, Chamkha studied unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption [21]. Keeping in view the wide area of practical importance of multi-fluid flows as mentioned above, it is the objective of the present study to investigate unsteady flow and heat transfer of two immiscible fluids in a horizontal channel with transpiring walls with time-dependent oscillating normal velocity.

1. Mathematical Formulation

The physical configuration Fig. 1 consists of two infinitely long, parallel and permeable plates maintained at different constant temperatures, extending in the Z and X directions. The region $0 \leq y \leq h$ (Region-I) is filled with a viscous fluid having density ρ_1, dynamic viscosity μ_1, specific heat at constant pressure Cp_1 and thermal conductivity K_1. The region $-h \leq y \leq 0$ (Region-II) is filled with a different viscous fluid having density ρ_2, dynamic viscosity μ_2, specific heat at constant pressure Cp_2 and thermal conductivity K_2.

It is assumed that the flow is unsteady, fully developed and that all fluid properties are constant. The flow in both regions is assumed to be driven by a common pressure gradients ($(-\partial p/\partial x)$) and temperature gradients $\Delta T = T_{w1} - T_{w2}$, where T_{w1} is the temperature of the wall boundary at $y = h$ and T_{w2} is the temperature of the wall boundary at $y = -h$.

Under these assumptions and considering $\rho_1 = \rho_2 = \rho_0$ and $Cp_1 = Cp_2 = Cp$, the governing equations of motion and energy [10] are formulated:

Region-I

$$\frac{\partial v_1}{\partial y} = 0, \tag{1}$$

$$\rho_0 \left(\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial y} \right) = \mu_1 \frac{\partial^2 u_1}{\partial y^2} - \frac{\partial p}{\partial x}, \tag{2}$$

$$\rho_0 Cp \left(\frac{\partial T_1}{\partial t} + v_1 \frac{\partial T_1}{\partial y} \right) = K_1 \frac{\partial^2 T_1}{\partial y^2} + \mu_1 \left(\frac{\partial u_1}{\partial y} \right)^2, \tag{3}$$

Region-II

$$\frac{\partial v_2}{\partial y} = 0, \tag{4}$$

$$\rho_0 \left(\frac{\partial u_2}{\partial t} + v_2 \frac{\partial u_2}{\partial y} \right) = \mu_2 \frac{\partial^2 u_2}{\partial y^2} - \frac{\partial p}{\partial x}, \tag{5}$$

$$\rho_0 Cp \left(\frac{\partial T_2}{\partial t} + v_2 \frac{\partial T_2}{\partial y} \right) = K_2 \frac{\partial^2 T_2}{\partial y^2} + \mu_2 \left(\frac{\partial u_2}{\partial y} \right)^2, \tag{6}$$

where u is the x-component of fluid velocity, v is the y-component of fluid velocity and T is the fluid temperature. The subscripts 1 and 2 correspond to region-I and region-II, respectively. The boundary conditions on velocity are the no-slip boundary conditions which require the x-component of velocity to vanish at the wall. The boundary conditions on temperature are the isothermal conditions. We also assume continuity of velocity, shear stress, temperature and heat flux at the interface between the two fluid layers at $y = 0$.

The hydrodynamic boundary and interface conditions for the two fluids can then be written as

$$\begin{gathered} u_1(h) = 0, \qquad u_2(-h) = 0, \qquad u_1(0) = u_2(0), \\ \mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y} \qquad \text{at} \qquad y = 0. \end{gathered} \tag{7}$$

The thermal boundary and interface conditions on temperature for both fluids are given by:

$$
\begin{gathered}
T_1(h) = T_{w_1}, \qquad T_2(-h) = T_{w_2}, \qquad T_1(0) = T_2(0), \\
K_1 \frac{\partial T_1}{\partial y} = K_2 \frac{\partial T_2}{\partial y} \qquad \text{at} \qquad y = 0.
\end{gathered} \tag{8}
$$

The continuity equations of both fluids (Eqs (1), (4)) imply that, v_1 and v_2 are independent of y, they can be utmost functions of time alone. Hence, we can write (assuming $v_1 = v_2 = v$):

$$
v = v_0(1 + \varepsilon A e^{i\omega t}), \tag{9}
$$

where A is a real positive constant, ω is the frequency parameter and ε is a small constant such that $\varepsilon A \leq 1$. Here it is assumed that the transpiration velocity varies periodically with time about a non-zero constant mean v_0 (Sturat, 1955). When $\varepsilon A = 0$, the case of constant transpiration velocity is recovered. Using the following non-dimensional quantities:

$$
\begin{gathered}
u_i = \overline{u}_1 u_i^*, \qquad y = h y^*, \qquad t = \frac{h^2}{\nu} t^*, \qquad v = \frac{\nu}{h} v^* = \frac{v}{|v_0|}, \qquad \omega = \frac{\nu}{h^2}\omega^*, \\
P = \frac{h^2}{\mu_1 \overline{u}_1}\left(\frac{\partial P}{\partial x}\right), \qquad \theta = \frac{T - T_w}{\overline{u}_1 \mu_1 / k_1}
\end{gathered} \tag{10}
$$

and for simplicity dropping the asterisks, Eqs (2) – (6) become:

Region-I

$$
\frac{\partial u_1}{\partial t} + v \frac{\partial u_1}{\partial y} = P + \frac{\partial^2 u_1}{\partial y^2}, \tag{11}
$$

$$
\frac{\partial \theta_1}{\partial t} + v \frac{\partial \theta_1}{\partial y} = \frac{1}{\mathrm{Pr}} \frac{\partial^2 \theta_1}{\partial y^2} + \frac{1}{\mathrm{Pr}} \left(\frac{\partial u_1}{\partial y}\right)^2, \tag{12}
$$

Region-II

$$
\frac{\partial u_2}{\partial t} + v \frac{\partial u_2}{\partial y} = P + \alpha \frac{\partial^2 u_2}{\partial y^2}, \tag{13}
$$

$$
\frac{\partial \theta_2}{\partial t} + v \frac{\partial \theta_2}{\partial y} = \frac{\beta}{\mathrm{Pr}} \frac{\partial^2 \theta_2}{\partial y^2} + \frac{\alpha}{\mathrm{Pr}} \left(\frac{\partial u_2}{\partial y}\right)^2, \tag{14}
$$

where $\mathrm{Pr} = \rho \nu C p / K_1$ is the Prandtl number; $\alpha = \mu_2/\mu_1$ is the ratio of dynamic viscosities and $\beta = K_2/K_1$ is the ratio of thermal conductivities.

The hydrodynamic and thermal boundary and interface conditions for both fluids in non-dimensional form become

$$
u_1(1) = 0, \qquad u_2(-1) = 0, \qquad u_1(0) = u_2(0), \qquad \frac{\partial u_1}{\partial y} = \alpha \frac{\partial u_2}{\partial y} \qquad \text{at} \qquad y = 0, \tag{15}
$$

$$
\theta_1(1) = 1, \qquad \theta_2(-1) = 0, \qquad \theta_1(0) = \theta_2(0), \qquad \frac{\partial \theta_1}{\partial y} = \beta \frac{\partial \theta_2}{\partial y} \qquad \text{at} \qquad y = 0. \tag{16}
$$

2. Closed-Form Solutions

The governing momentum Eqs (11) and (13) along with the energy Eqs (12) and (14) are solved subject to the boundary and interface conditions Eqs (15) and (16) for the velocity and temperature distributions in both regions. These equations are coupled partial differential equations that can not be solved in a closed form. However, it can be reduced to ordinary differential equations by assuming:

$$u_1(y,t) = u_{10}(y) + \varepsilon e^{i\omega t} u_{11}(y), \quad u_2(y,t) = u_{20}(y) + \varepsilon e^{i\omega t} u_{21}(y), \tag{17}$$

$$\theta_1(y,t) = \theta_{10}(y) + \varepsilon e^{i\omega t} \theta_{11}(y), \qquad \theta_2(y,t) = \theta_{20}(y) + \varepsilon e^{i\omega t} \theta_{21}(y). \tag{18}$$

This is a valid assumption because of the choice of v as defined in Eq. (9) that the amplitude $A\varepsilon \ll 1$.

Considering the real part of $e^{i\omega t}$, Eqs (17) and (18) become:

$$u_1(y,t) = u_{10}(y) + \varepsilon \cos\omega t \ u_{11}(y), \quad u_2(y,t) = u_{20}(y) + \varepsilon \cos\omega t \ u_{21}(y), \tag{19}$$

$$\theta_1(y,t) = \theta_{10}(y) + \varepsilon \cos\omega t \ \theta_{11}(y), \qquad \theta_2(y,t) = \theta_{20}(y) + \varepsilon \cos\omega t \ \theta_{21}(y). \tag{20}$$

By substitution of Eqs (19) and (20) into Eqs (11) to (16), keeping the harmonic and non-harmonic terms and neglecting the higher order terms of ε^2, one obtains following pairs of equations:

Region-I:

non-periodic coefficients

$$\frac{d^2u_{10}}{dy^2} - \frac{du_{10}}{dy} + P = 0, \tag{21}$$

$$\frac{d^2\theta_{10}}{dy^2} - \mathrm{Pr}\frac{d\theta_{10}}{dy} = -\left(\frac{du_{10}}{dy}\right)^2, \tag{22}$$

periodic coefficients

$$\frac{d^2u_{11}}{dy^2} - \frac{du_{11}}{dy} + \omega\tan\omega t \ u_{11} = A\frac{du_{10}}{dy}, \tag{23}$$

$$\frac{d^2\theta_{11}}{dy^2} - \mathrm{Pr}\frac{d\theta_{11}}{dy} + \mathrm{Pr}\ \omega\tan\omega t \ \theta_{11} = \mathrm{Pr}A\frac{d\theta_{10}}{dy} - 2\frac{du_{10}}{dy}\frac{du_{11}}{dy}. \tag{24}$$

Region-II:

non-periodic coefficients

$$\alpha\frac{d^2u_{20}}{dy^2} - \frac{du_{20}}{dy} + P = 0, \tag{25}$$

$$\beta\frac{d^2\theta_{20}}{dy^2} - \mathrm{Pr}\frac{d\theta_{20}}{dy} = -\alpha\left(\frac{du_{20}}{dy}\right)^2, \tag{26}$$

periodic coefficients

$$\alpha\frac{d^2\ u_{21}}{dy^2} - \frac{du_{21}}{dy} + \omega\tan\omega t u_{21} = A\frac{du_{20}}{dy}, \tag{27}$$

$$\beta \frac{d^2\theta_{21}}{dy^2} - \mathrm{Pr}\frac{d\theta_{21}}{dy} + \mathrm{Pr}\ \omega \tan \omega t \theta_{21} = \mathrm{Pr} A \frac{d\theta_{20}}{dy} - 2\alpha \frac{du_{20}}{dy}\frac{du_{21}}{dy}. \tag{28}$$

The corresponding boundary and interface conditions become as follows:

Non-periodic coefficients

$$\begin{gathered} u_{10}(1) = 0, \qquad u_{20}(-1) = 0, \qquad u_{10}(0) = u_{20}(0), \\ \frac{\partial u_{10}}{\partial y} = \alpha \frac{\partial u_{20}}{\partial y} \qquad \text{at} \qquad y = 0. \end{gathered} \tag{29}$$

Periodic coefficients

$$\begin{gathered} u_{11}(1) = 0, \qquad u_{21}(-1) = 0, \qquad u_{11}(0) = u_{21}(0), \\ \frac{\partial u_{11}}{\partial y} = \alpha \frac{\partial u_{21}}{\partial y} \qquad \text{at} \qquad y = 0. \end{gathered} \tag{30}$$

Non-periodic coefficients

$$\begin{gathered} \theta_{10}(1) = 1, \qquad \theta_{20}(-1) = 0, \qquad \theta_{10}(0) = \theta_{20}(0), \\ \frac{\partial \theta_{10}}{\partial y} = \beta \frac{\partial u_{20}}{\partial y} \qquad \text{at} \qquad y = 0. \end{gathered} \tag{31}$$

Periodic coefficients

$$\begin{gathered} \theta_{11}(1) = 0, \qquad \theta_{21}(-1) = 0, \qquad \theta_{11}(0) = \theta_{21}(0), \\ \frac{\partial \theta_{11}}{\partial y} = \beta \frac{\partial \theta_{21}}{\partial y} \quad \text{at} \quad y = 0. \end{gathered} \tag{32}$$

Eqs (21) – (28) along with the boundary and interface conditions Eqs (29) – (32) represent a system of ordinary differential equations and conditions that can be solved in a closed form.

The solutions of non-periodic (harmonic) terms lead to the steady flow solutions for both fluids. Without going into detail, the steady-state velocity and temperature profiles can be shown in a following form:

$$u_{10} = C_1 + C_2 e^{y}, \tag{33}$$

$$u_{20} = C_3 + C_4 e^{m_1 y}, \tag{34}$$

$$\theta_{10} = C_5 + C_6 e^{\mathrm{Pr} y} + k_1 e^{2y} + k_2 e^{y} + k_3 y, \tag{35}$$

$$\theta_{20} = C_7 + C_8 e^{m_2 y} + k_4 e^{2 m_1 y} + k_5 e^{m_1 y} + k_6 y. \tag{36}$$

On the other hand, the solutions of periodic (non-harmonic) terms or transient velocity and temperature profiles in both regions of the channel (Region-I and Region-II) take on different forms depending on the value of $4\omega \tan \omega t$ for both velocity and temperature. These forms or cases can be shown to be the following ones:

Case-I

Region-I

$$u_{11} = C_9 e^{m_3 y} + C_{10} e^{m_4 y} + \tfrac{A}{k_7}(C_2 e^y + P)$$
$$\text{for} \qquad 4\omega \tan\omega t < 1, \tag{37}$$

$$\theta_{11} = C_{13} e^{m_7 y} + C_{14} e^{m_8 y} + k_{10} e^{\mathrm{Pr} y} + k_{14} e^{m_9 y} + k_{15} e^{m_{10} y}$$
$$+ k_{16} e^{m_3 y} + k_{17} e^{m_4 y} + k_{20} e^{2y} + k_{21} e^y + k_{13} \tag{38}$$
$$\text{for} \qquad 4\omega \tan\omega t < \mathrm{Pr}.$$

Region-II

$$u_{21} = C_{11} e^{m_5 y} + C_{12} e^{m_6 y} + k_8 e^{m_1 y} + k_9$$
$$\text{for} \qquad 4\alpha\omega \tan\omega t < 1, \tag{39}$$

$$\theta_{21} = C_{15} e^{m_{11} y} + C_{16} e^{m_{12} y} + k_{22} e^{m_2 y} + k_{33} e^{2 m_1 y} + k_{26} e^{m_{13} y}$$
$$+ k_{27} e^{m_{14} y} + k_{28} e^{m_5 y} + k_{29} e^{m_6 y} + k_{32} e^{m_1 y} + k_{25} \tag{40}$$
$$\text{for} \qquad 4\beta\omega \tan\omega t < \mathrm{Pr}.$$

Case-II

Region-I

$$u_{11} = (D_1 + D_2 y) e^{J_1 y} + \tfrac{A}{k_7}(C_2 e^y + P)$$
$$\text{for} \qquad 4\omega \tan\omega t = 1, \tag{41}$$

$$\theta_{11} = (D_5 + D_6 y) e^{J_3 y} + P_{20} e^{n_1 y} + P_{19} y e^{n_1 y} + P_{23} e^{J_1 y}$$
$$+ P_{22} y e^{J_3 y} + P_{26} e^{2y} + P_{27} e^y + k_{10} e^{\mathrm{Pr} y} + k_{13} \tag{42}$$
$$\text{for} \qquad 4\omega \tan\omega t = \mathrm{Pr}.$$

Region-II

$$u_{11} = (D_3 + D_4 y) e^{J_2 y} + k_8 e^{m_1 y} + k_9$$
$$\text{for} \qquad 4\alpha\omega \tan\omega t = 1, \tag{43}$$

$$\theta_{21} = (D_7 + D_8 y) e^{J_4 y} + P_{31} e^{n_2 y} + P_{32} y e^{n_2} y + P_{34} e^{J_2 y}$$
$$+ P_{35} y e^{J_2 y} + P_{38} e^{2 m_1 y} + P_{39} e^{m_1 y} + k_{22} e^{m_2 y} + k_{25} \tag{44}$$
$$\text{for} \qquad 4\beta\omega \tan\omega t = \mathrm{Pr}.$$

Case-III

Region-I

$$u_{11} = e^{\gamma_1 y}(G_1 \cos\delta_1 y + G_2 \sin\delta_1 y) + \frac{A}{k_7}(C_2 e^y + P)$$
$$\text{for} \qquad 4\omega \tan\omega t > 1, \tag{45}$$

$$\theta_{11} = e^{\gamma_3 y}(G_5 \cos\delta_3 y + G_6 \sin\delta_3 y) + k_{10}e^{\mathrm{Pr}y} + V_{12}e^{2y} + V_{13}e^{y}$$
$$+V_6 e^{w_1 y}(V_4 cos\delta_1 y - V_5 \sin\delta_1 y) + V_9 e^{\gamma_1 y}(V_7 cos\delta_1 y - V_8 \sin\delta_1 y) + k_{13} \tag{46}$$
$$\text{for} \quad 4\omega\tan\omega t > \mathrm{Pr}.$$

Region-II

$$u_{21} = e^{\gamma_2 y}(G_3 \cos\delta_2 y + G_4 \sin\delta_2 y) + k_8 e^{m_1 y} + k_9$$
$$\text{for} \quad 4\alpha\omega\tan\omega t > 1, \tag{47}$$

$$\theta_{21} = e^{\gamma_4 y}(G_7 \cos\delta_4 y + G_8 \sin\delta_4 y) + k_{22}e^{m_2 y} + V_{24}e^{2m_1 y} + V_{25}e^{m_1 y}$$
$$+V_{18}e^{w_2 y}(V_{16}\cos\delta_2 y - V_{17}\sin\delta_2 y) + V_{21}e^{\gamma_2 y}(V_{19}\cos\delta_2 y - V_{20}\sin\delta_2 y) + k_{25} \tag{48}$$
$$\text{for} \quad 4\beta\omega\tan\omega t > \mathrm{Pr}.$$

It should be noted that all of the constants appearing in the above solutions are defined in the Appendix section.

3. Results and Discussion

In this section, representative flow and heat transfer results for unsteady oscillatory flow of two immiscible fluids through a horizontal channel with time-dependent transpiration velocity are presented and discussed for various parametric conditions. Although exact solutions were obtained for the steady-state conditions with constant transpiration velocity, analytical solutions for variable transpiration velocity were obtained under the constraint that the coefficient of cosine of periodic frequency parameter ε is small as shown in Eqs (17), (18). The results are shown for small, comparable and large values of frequency parameter and Prandtl number. The flow solutions are depicted graphically in Figs 2 – 10 for different values of the governing parameters such as viscosity ratio α, conductivity ratio β and Prandtl number Pr. The effects of amplitude εA and periodic frequency ωt on the velocity and temperature profiles in both regions of the channel are shown in Tables 1 – 6.

Fig. 2 presents typical velocity profiles in the channel for different values of the viscosity ratio α. As the viscosity ratio increases, the velocity profiles are suppressed for frequency parameter less than inverse tangential periodic frequency parameter ($4\omega\tan\omega t < 1$). Another words, as the fluid in the lower region becomes thick and hence, the velocity decreases. The temperature profiles are also suppressed as the viscosity ratio increases as shown in Fig. 3. The magnitude of suppression is less in Region-II compared to Region-I. Fig. 4 depicts the effect of thermal conductivity ratio β on the temperature profiles. $As\beta$ increases, and the magnitude of suppression is larger in Region-II compared to Region-I. This is obvious because the lower plate is maintained at a lower temperature compared to Region-I. The effect of Prandtl number on temperature profiles is shown in Fig. 5. The Prandtl number is a measure of momentum diffusion to heat diffusion. It is a measure of the relative importance of viscosity and heat conduction in a flow field. Thus, as the Prandtl number increases, the viscous forces dominate over heat conduction and hence, the temperature decreases. Figs 3 – 5 are drawn for a frequency parameter less than Prandtl number ($4\omega\tan\omega t < \mathrm{Pr}$).

Typical temperature profiles are shown in Fig. 6 for a comparable frequency parameter and Prandtl number ($4\omega\tan\omega t = \mathrm{Pr}$). As the frequency coefficient ω increases, the temperature increases. However, this increase in temperature is more rapid for small values of ω compared to

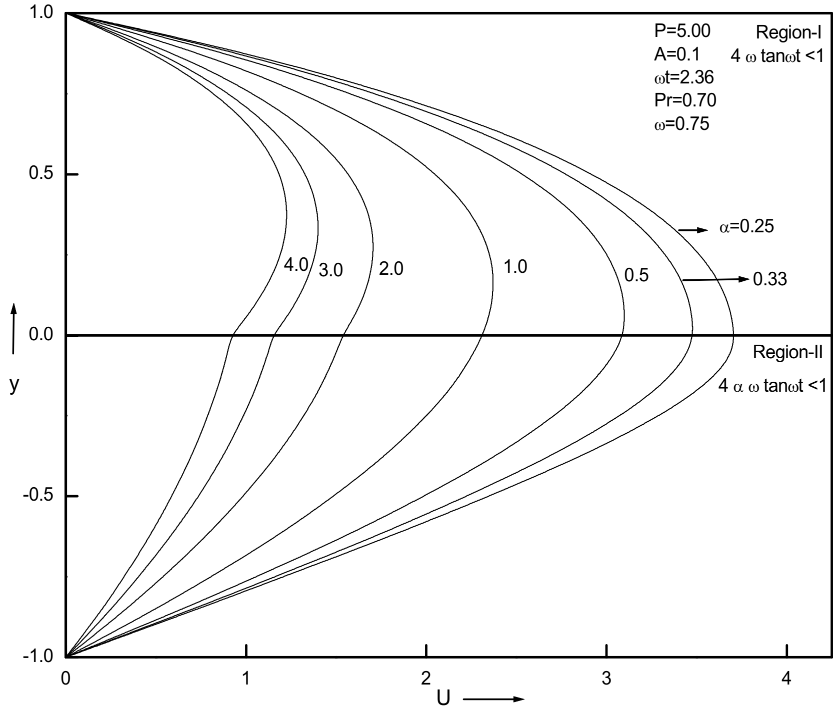

Fig. 2. Velocity profiles for different values of ratio of viscosity α.

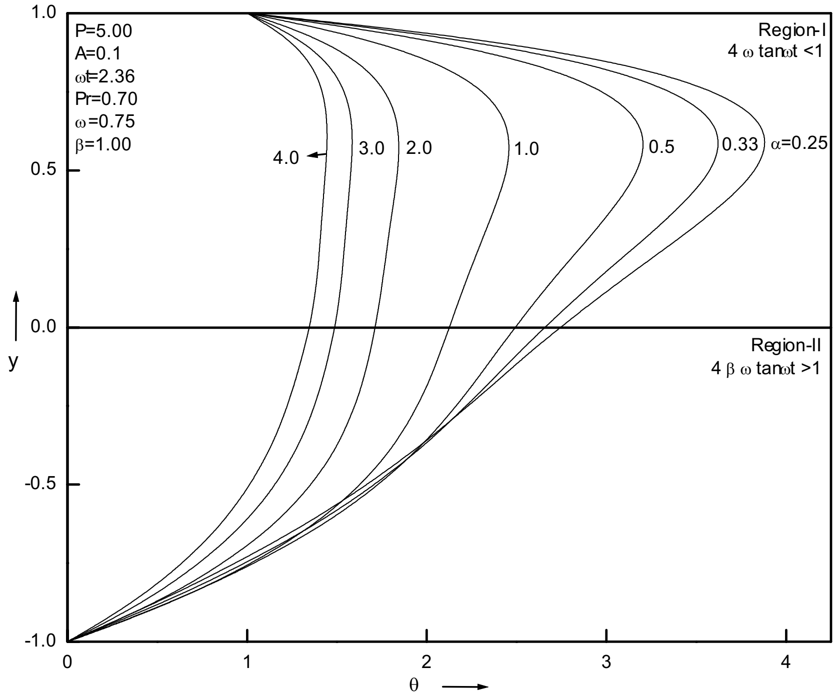

Fig. 3. Temperature profiles for different values of ratio of viscosity α.

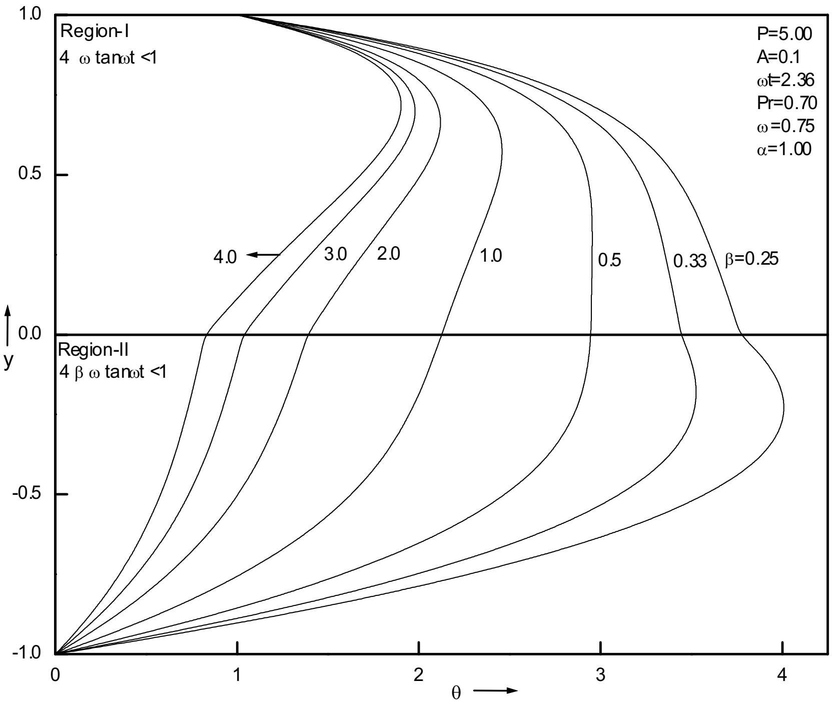

Fig. 4. Temperature profiles for different values of thermal conductivity β.

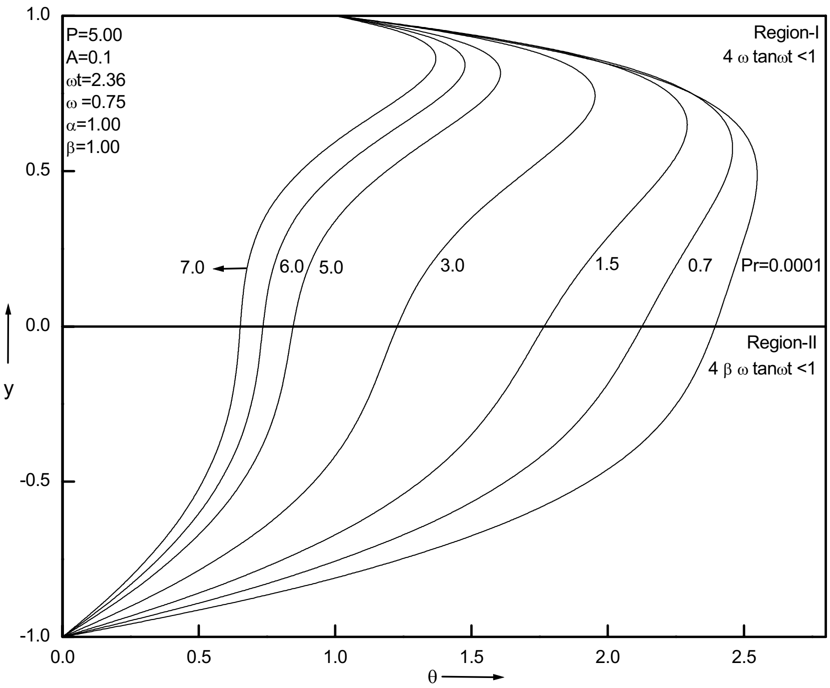

Fig. 5. Temperature profiles for different values of Prandtl number Pr.

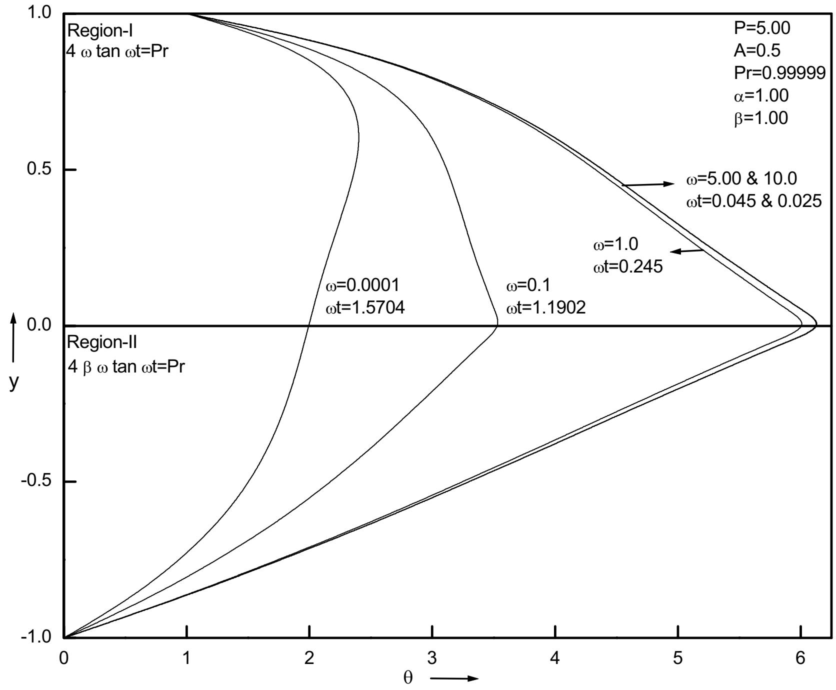

Fig. 6. Temperature profiles for different values of frequency parameter ω and periodic frequency parameter ωt.

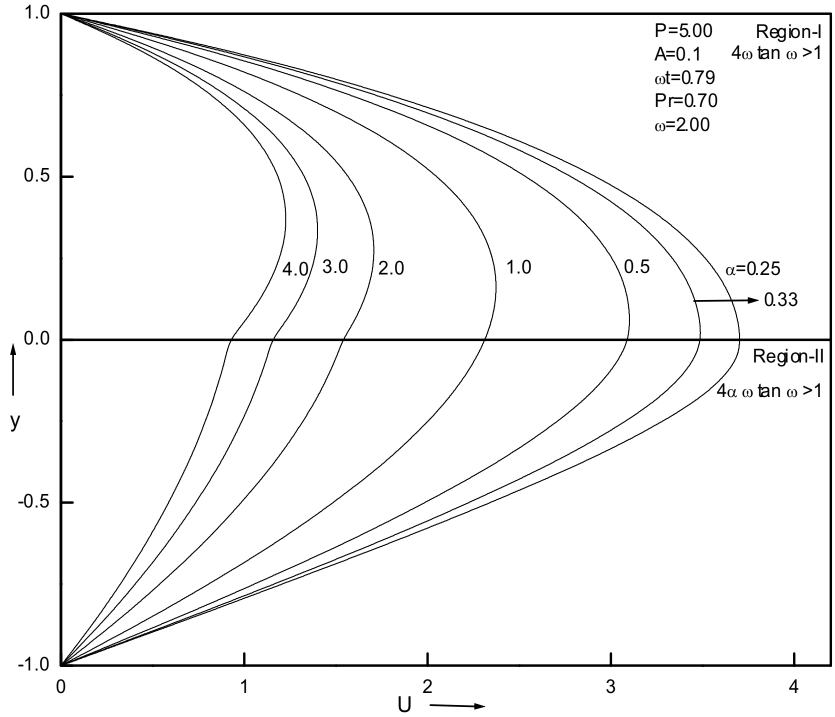

Fig. 7. Velocity profiles for different values of ratio of viscosity α.

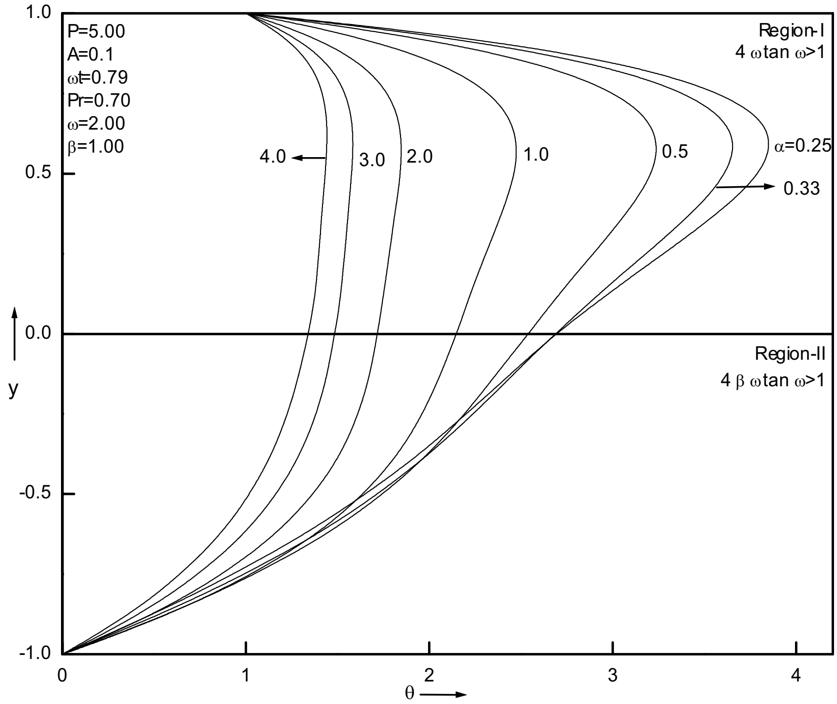

Fig. 8. Temperature profiles for different values of ratio of viscosity α.

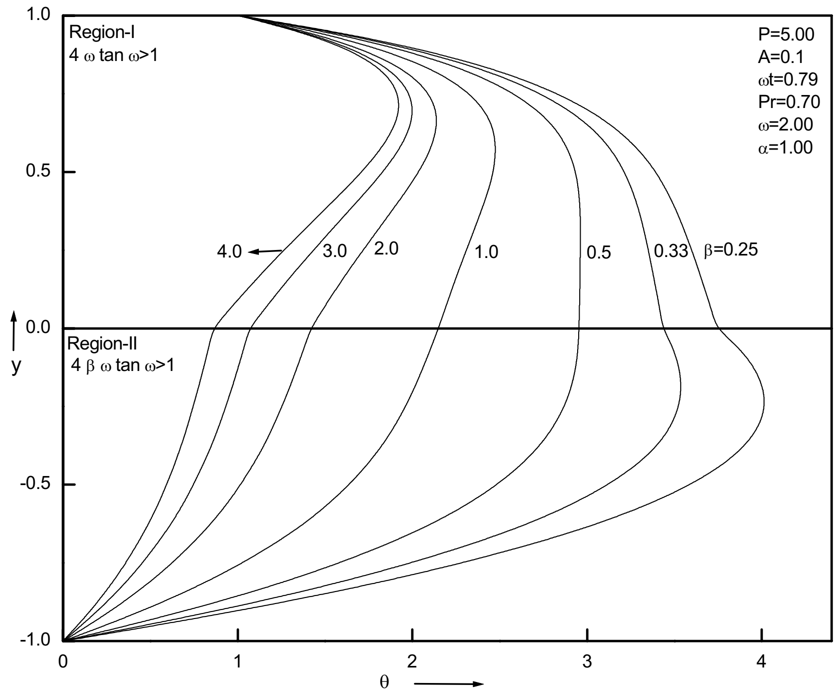

Fig. 9. Temperature profiles for different values of thermal conductivity β.

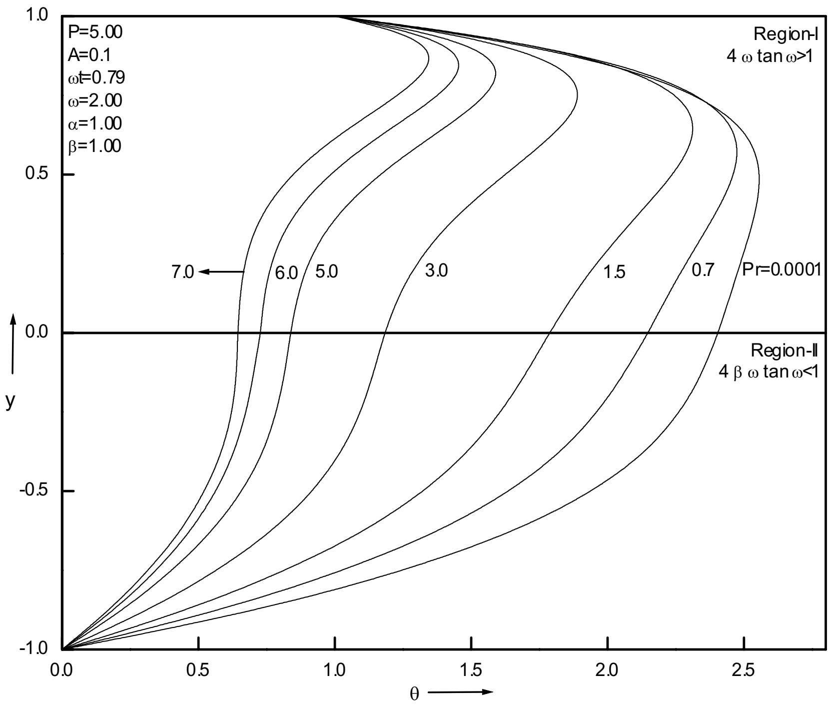

Fig. 10. Temperature profiles for different values of Prandtl number Pr.

Table 1
Velocity and temperature for different values of amplitude
for $\omega = 0.75$, $\omega t = 0.3217$ and $\mathrm{Pr} = 0.7$.

	$\varepsilon A = 0.00$		$\varepsilon A = 0.10$		$\varepsilon A = 0.20$	
y	U	θ	U	θ	U	θ
1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
0.80	1.0964	2.1929	1.0852	2.1952	1.0740	2.1976
0.60	1.8128	2.4589	1.8036	2.4773	1.7944	2.4956
0.40	2.2181	2.3909	2.2174	2.4187	2.2166	2.4465
0.20	2.3686	2.2532	2.3783	2.2840	2.3880	2.3147
0.00	2.3106	2.1253	2.3295	2.1571	2.3485	2.1889
-0.20	2.0818	1.9891	2.1070	2.0233	2.1321	2.0575
-0.40	1.7132	1.7822	1.7402	1.8197	1.7672	1.8571
-0.60	1.2302	1.4285	1.2539	1.4663	1.2776	1.5040
-0.80	0.6535	0.8550	0.6683	0.8833	0.6831	0.9115
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2
Velocity and temperature for different values of frequency parameter
for $\omega = 0.75$, $\varepsilon A = 0.001$ and $\mathrm{Pr} = 0.7$.

	$\omega t = 2.0952$		$\omega t = 2.3571$		$\omega t = 2.6190$	
y	U	θ	U	θ	U	θ
1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
0.80	1.0963	2.1929	1.0964	2.1929	1.0963	2.1929
0.60	1.8127	2.4590	1.8128	2.4589	1.8127	2.4592
0.40	2.2181	2.3910	2.2181	2.3909	2.2181	2.3913
0.20	2.3687	2.2534	2.3686	2.2532	2.3687	2.2537
0.00	2.3107	2.1255	2.3106	2.1253	2.3108	2.1258
-0.20	2.0820	1.9893	2.0818	1.9891	2.0821	1.9896
-0.40	1.7134	1.7824	1.7132	1.7822	1.7136	1.7827
-0.60	1.2304	1.4288	1.2302	1.4285	1.2305	1.4290
-0.80	0.6536	0.8552	0.6535	0.8550	0.6537	0.8554
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

larger values of ω and there will be no change in temperature for $\omega > 5$ and onwards. This behavior can clearly be seen in Fig. 6.

Fig. 7 depicts the velocity profiles for frequency parameter greater than the inverse tangential periodic frequency parameter ($4\omega \tan \omega t > 1$). As the viscosity ratio α increases, the flow velocity in the channel decreases which is a similar result for the case when $4\omega \tan \omega t < 1$. Figs 8 – 10 show the temperature profiles for a frequency parameter greater than the Prandtl number ($4\omega \tan \omega t > \mathrm{Pr}$). As the viscosity ratio increases, the flow field is suppressed and its velocity decreases as shown in Fig. 8. The magnitude of suppression is large in Region-I compared to Region-II. The temperature decreases as the conductivity ratio β and Prandtl number increase which is the same result as the obtained for the case when $4\omega \tan \omega t < \mathrm{Pr}$.

Tables 1 – 6 display the effects of the amplitude and periodic frequency parameter on the flow field. Tables 1, 2 show these effects for the case when $4\omega \tan \omega t < 1$, Tables 3, 4 show them for the case when $4\omega \tan \omega t = 1$ and Tables 5 and 6 present them for the case when $4\omega \tan \omega t > 1$, respectively. It is seen that as the amplitude εA increases, the flow velocity also increases for all the cases but the magnitude is largest for the case when $4\omega \tan \omega t = 1$. The effect on the temperature field due to the increase in amplitude is such that it also increases as εA increases. The magnitude of this growth is very large for a comparable frequency parameter, which is observed in Tables 1, 3, 5.

Table 4 displays the effect of the frequency parameter on the flow velocity for case when $4\omega \tan \omega t = 1$. Its effect on temperature is shown in Fig. 6. The velocity increases slightly for $\omega < 1$ and remains almost constant for $\omega > 1$.

The effect of the periodic frequency parameter ωt on the velocity and temperature values is shown in Tables 2, 6. As the periodic frequency parameter ωt increases, both the velocity and temperature remain almost constant in the case when $4\omega \tan \omega t < 1$ but they decrease in the case when $4\omega \tan \omega t > 1$.

Thus, one can conclude that the flow can be controlled by considering different fluids having different viscosities, conductivities and also by varying the oscillation amplitude.

Table 3
Velocity and temperature for different values of amplitude
for $\omega = 0.75$, $\omega t = 0.3217$ and $\mathrm{Pr} = 0.99999$.

	$\varepsilon A = 0.00$		$\varepsilon A = 0.10$		$\varepsilon A = 0.20$	
y	U	θ	U	θ	U	θ
1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
0.80	1.0964	2.1820	1.4211	17.3751	1.7457	32.5728
0.60	1.8128	2.4040	2.3179	33.0554	2.8230	63.715
0.40	2.2181	2.2966	2.7856	48.9534	3.3531	95.6214
0.20	2.3686	2.1331	2.9018	65.3874	3.4350	128.6551
0.00	2.3106	1.9913	2.7304	82.3613	3.1502	162.7464
-0.20	2.0818	1.8507	2.3676	62.1130	2.6533	122.3864
-0.40	1.7132	1.6492	1.8919	44.0401	2.0705	86.4385
-0.60	1.2302	1.3150	1.3269	27.8298	1.4236	54.3492
-0.80	0.6535	0.7826	0.6913	13.2213	0.7292	25.662
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4
Velocity for different values of ω and ωt
for $\varepsilon A = 0.005$ and $\mathrm{Pr} = 0.99999$.

y	$\omega = 0.0001$	$\omega = 0.1$	$\omega = 1.0$	$\omega = 5.0$	$\omega = 10.0$
	$\omega t = 1.5704$	$\omega t = 1.1902$	$\omega t = 0.245$	$\omega t = 0.045$	$\omega t = 0.0245$
1.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.80	1.0964	1.1028	1.1130	1.1135	1.1135
0.60	1.8128	1.8227	1.8386	1.8394	1.8394
0.40	2.2181	2.2292	2.2471	2.2479	2.2480
0.20	2.3686	2.3790	2.3959	2.3967	2.3967
0.00	2.3106	2.3188	2.3321	2.3327	2.3327
-0.20	2.0818	2.0874	2.0964	2.0969	2.0969
-0.40	1.7132	1.7167	1.7224	1.7226	1.7227
-0.60	1.2302	1.2321	1.2352	1.2353	1.2353
-0.80	0.6535	0.6542	0.6554	0.6555	0.6555
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000

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Table 5
Velocity and temperature for different values amplitude
for $\omega = 2.0$, $\omega t = 0.7857$ and $\mathrm{Pr} = 0.7$.

	$\varepsilon A = 0.00$		$\varepsilon A = 0.10$		$\varepsilon A = 0.20$	
y	U	θ	U	θ	U	θ
1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
0.80	1.0964	2.1911	1.2070	3.3200	1.3176	4.4488
0.60	1.8128	2.4557	2.0018	4.3409	2.1909	6.2262
0.40	2.2181	2.3865	2.4546	4.7636	2.6912	7.1406
0.20	2.3686	2.2482	2.6239	4.9114	2.8793	7.5747
0.00	2.3106	2.1199	2.5590	4.8926	2.8075	7.6652
-0.20	2.0818	1.9857	2.2672	3.8246	2.4527	5.6635
-0.40	1.7132	1.7801	1.8376	3.0473	1.9619	4.3144
-0.60	1.2302	1.4274	1.3010	2.2692	1.3719	3.1111
-0.80	0.6535	0.8546	0.6822	1.2928	0.7109	1.7311
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 6
Velocity and temperature for different values of frequency parameter
for $\omega = 2.0$, $\varepsilon A = 0.001$ and $\mathrm{Pr} = 0.7$.

	$\omega t = 0.5238$		$\omega t = 0.7857$		$\omega t = 1.0476$	
y	U	θ	U	θ	U	θ
1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
0.80	1.0977	2.1940	1.0975	2.2024	1.0729	1.6294
0.60	1.8149	2.4591	1.8147	2.4746	1.7687	1.4934
0.40	2.2207	2.3895	2.2204	2.4103	2.1589	1.1815
0.20	2.3713	2.2504	2.3712	2.2748	2.3018	0.9469
0.00	2.3132	2.1213	2.3131	2.1477	2.2446	0.8529
-0.20	2.0836	1.9899	2.0837	2.0040	2.0210	1.4933
-0.40	1.7144	1.7861	1.7145	1.7928	1.6643	1.6365
-0.60	1.2308	1.4333	1.2309	1.4358	1.1970	1.4038
-0.80	0.6537	0.8585	0.6538	0.8590	0.6374	0.8527
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

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Appendix

$$C_1 = -(P + C_2 e^1); \qquad C_2 = \frac{2P - Pe^{-m_1-1} - \alpha P}{e^{-m_1} - e^1}; \qquad C_3 = C_1 + C_2 - C_4;$$

$$C_4 = C_2 + P - \alpha P; \qquad C_5 = 1 - C_6 e^{\mathrm{Pr}} - l_1; \qquad C_6 = \frac{l_1 - l_6 - l}{l_5 - e^{\mathrm{Pr}}};$$

$$C_7 = -C_8 e^{-m_2} - l_2; \qquad C_8 = \frac{\mathrm{Pr} C_6 + l_4}{\beta m_2}; \qquad C_9 = \frac{l_{18} l_7 - l_{19} e^{m_4}}{l_{17} e^{m_4} - l_{18} e^{m_3}};$$

$$C_{10} = -\frac{C_9 e^{m_3} + l_7}{e_{m_4}}; \qquad C_{11} = -\frac{C_9 + C_{10} + l_{16}}{l_{15}}; \qquad C_{12} = -\frac{C_{11} e^{-m_5} + l_8}{e^{-m_6}};$$

$$C_{13} = \frac{l_{31} l_{20} - l_{32} e^{m_8}}{l_{30} e^{m_8} - l_{31} e^{m_7}}; \qquad C_{14} = -\frac{C_{13} e^{m_7} + l_{20}}{e^{m_8}}; \qquad C_{15} = -\frac{C_{13} + C_{14} + l_{29}}{l_{28}};$$

$$C_{16} = \frac{m_7 C_{13} + m_8 C_{14} - \beta m_{11} C_{15} + l_{23}}{\beta m_{12}};$$

$$D_1 = \frac{P_2P_{13} - P_3}{P_1 - P_2P_{12}}; \qquad D_2 = -\frac{P_1D_1 + P_3}{P_2}; \qquad D_3 = D_1 + P_7;$$

$$D_4 = \frac{-(P_4D_3 + P_6)}{P_5}; \qquad D_5 = \frac{Q_1 - e^{J_3}Q_8}{e^{n_3}Q_8 - e^{J_3}}; \qquad D_6 = -D_5 - \frac{Q_1}{e^{J_3}};$$

$$D_7 = Q_3 + D_5; \qquad D_8 = D_7 - \frac{Q_2}{e^{-J_4}};$$

$$F_1 = 2m_1 - \mathrm{Pr}; \qquad F_2 = n_1^2 - \mathrm{Pr}n_1 + \mathrm{Pr}k_7; \qquad F_3 = 2J_1 - \mathrm{Pr};$$

$$F_4 = J_1^2 - \mathrm{Pr}J_1 + \mathrm{Pr}k_7; \qquad F_5 = 2\beta n_2 - \mathrm{Pr}; \qquad F_6 = \beta n_2^2 - \mathrm{Pr}n_2 + \mathrm{Pr}k_7;$$

$$G_1 = \frac{R_3\delta_2 - R_2R_{13}}{R_{12}R_2 - R_1\delta_2}; \qquad G_2 = -\frac{R_1G_1 + R_3}{R_2}; \qquad G_3 = G_1 + R_7;$$

$$G_4 = \frac{-(R_4G_3 + R_6)}{R_5}; \qquad G_5 = \frac{R_{16}R_{25} - R_{17}\delta_3}{R_{14}\delta_3 - R_{16}R_{25}}; \qquad G_6 = -\frac{R_{14}G_5 + R_{17}}{R_{16}};$$

$$G_7 = G_5 + R_{21}; \qquad G_8 = \frac{\gamma_3G_5 + \delta_3G_6 - \beta\delta_4G_7 + R_{22}}{\beta\delta_4};$$

$$J_1 = \frac{1}{2}; \qquad J_2 = \frac{1}{2\alpha}; \qquad J_3 = \frac{\mathrm{Pr}}{2}; \qquad J_4 = \frac{\mathrm{Pr}}{2\beta};$$

$$k_1 = -\frac{C_2^2}{4 - 2\mathrm{Pr}}; \qquad k_2 = -\frac{2C_2P}{1 - \mathrm{Pr}}; \qquad k_3 = \frac{P^2}{\mathrm{Pr}};$$

$$k_4 = -\frac{\alpha m_1^2C_4^2}{4\beta m_1^2 - 2\mathrm{Pr}m_1}; \qquad k_5 = -\frac{2\alpha m_1C_4P}{\beta m_1^2 - \mathrm{Pr}m_1}; \qquad k_6 = \frac{\alpha P^2}{\mathrm{Pr}};$$

$$k_7 = \omega\tan\omega t; \qquad k_8 = \frac{Am_1C_4}{\alpha m_1^2 - m_1 + k_7}; \qquad k_9 = \frac{AP}{k_7};$$

$$k_{10} = \frac{A\mathrm{Pr}C_6}{k_7}; \qquad k_{11} = \frac{2\mathrm{Pr}Ak_1}{4 - 2\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{12} = \frac{\mathrm{Pr}Ak_2}{1 - \mathrm{Pr} + \mathrm{Pr}k_7};$$

$$k_{13} = \frac{Ak_3}{k_7}; \qquad k_{14} = -\frac{2m_3C_2C_9}{m_9^2 - m_9\mathrm{Pr} + \mathrm{Pr}k_7};$$

$$k_{15} = -\frac{2m_3C_2C_{10}}{m_{10}^2 - m_{10}\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{16} = -\frac{2Pm_3C_9}{m_3^2 - m_3\mathrm{Pr} + \mathrm{Pr}k_7};$$

$$k_{17} = -\frac{2Pm_4C_{10}}{m_4^2 - m_4\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{18} = -\frac{2AC_2^2}{k_7(4 - 2\mathrm{Pr} + \mathrm{Pr}k_7)};$$

$$k_{19} = -\frac{2AC_2P}{k_7(1 - \mathrm{Pr} + \mathrm{Pr}k_7)}; \qquad k_{20} = k_{11} + k_{18}; \qquad k_{21} = k_{12} + k_{19};$$

$$k_{22} = \frac{\mathrm{Pr}Am_2C_8}{m_2^2\beta - m_2\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{23} = \frac{2\mathrm{Pr}Am_1k_4}{4m_1^2\beta - 2m_1\mathrm{Pr} + \mathrm{Pr}k_7};$$

$$k_{24} = \frac{\mathrm{Pr}Am_1k_5}{m_1^2\beta - m_1\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{25} = \frac{Ak_6}{k_7};$$

$$k_{26} = -\frac{2\alpha m_1m_5C_4C_{11}}{\beta m_{13}^2 - \mathrm{Pr}m_{13} + \mathrm{Pr}\,k_7}; \qquad k_{27} = -\frac{2\alpha m_1m_6C_4C_{12}}{\beta m_{14}^2 - \mathrm{Pr}m_{14} + \mathrm{Pr}k_7};$$

$$k_{28} = -\frac{2\alpha Pm_5C_{11}}{\beta m_5^2 - m_5\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{29} = -\frac{2\alpha Pm_6C_{12}}{\beta m_6^2 - m_6\mathrm{Pr} + \mathrm{Pr}\,k_7};$$

$$k_{30} = -\frac{2\alpha m_1^2C_4k_8}{4\beta m_1^2 - 2m_1\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad k_{31} = -\frac{2\alpha Pm_1k_8}{\beta m_1^2 - m_1\mathrm{Pr} + \mathrm{Pr}k_7};$$

$$k_{32} = k_{24} + k_{31}; \qquad k_{33} = k_{23} + k_{30};$$

$$l_1 = k_1e^2 + k_2e^1 + k_3; \qquad l_2 = k_4e^{-2m_1} + k_5e^{-m_1} - k_6; \qquad l_3 = k_1 + k_2 - k_4 - k_5;$$

$$l_4 = 2k_1 + k_2 + k_3 - 2\beta m_1k_4; \qquad l_5 = 1 + \frac{\mathrm{Pr}(e^{-m_2} - 1)}{m_2\beta};$$

$$l_6 = l_2 + l_3 + \frac{l_4(e^{-m_2} - 1)}{m_2\beta}; \qquad l_7 = \frac{A}{k_7}(C_2e^1 + P); \qquad l_8 = k_8e^{-m_1} + k_9;$$

$$l_9 = \frac{A}{k_7}(C_2 + P) - k_8 - k_9; \qquad l_{10} = \frac{A}{k_7}C_2 - \alpha m_1k_8; \qquad l_{11} = m_3 - \alpha m_6;$$

$$l_{12} = m_4 - \alpha m_6; \qquad l_{13} = \alpha m_6 - \alpha m_5; \qquad l_{14} = l_{10} - \alpha m_6l_9;$$

$$l_{15}=\frac{e^{-m_5}}{e^{-m_6}}-1;\qquad l_{16}=\frac{l_8}{e^{-m_6}}+l_9;\qquad l_{17}=l_{11}+\frac{l_{13}}{l_{15}};$$

$$l_{18}=l_{12}-\frac{l_{13}}{l_{15}};\qquad l_{19}=l_{14}-\frac{l_{13}l_{16}}{l_{15}};$$

$$l_{20}=k_{10}e^{\mathrm{Pr}}+k_{14}e^{m_9}+k_{15}e^{m_{10}}+k_{16}e^{m_3}+k_{17}e^{m_4}+k_{20}e^{2}+k_{21}e^{1}+k_{13};$$

$$l_{21}=k_{22}e^{-m_2}+k_{33}e^{-2m_1}+k_{26}e^{-m_{13}}+k_{27}e^{-m_4}+k_{28}e^{-m_5}+k_{29}e^{-m_6}+k_{32}e^{-m_1}+k_{25};$$

$$l_{22}=k_{10}+k_{14}+k_{15}+k_{16}+k_{17}+k_{20}+k_{21}+k_{13}-k_{22}-k_{33}-k_{26}-k_{27}-k_{28}-k_{29}-k_{32}-k_{25};$$

$$l_{23}=\mathrm{Pr}k_{10}+m_9k_{14}+m_{10}k_{15}+m_3k_{16}+m_4k_{17}+2k_{20}+k_{21}$$
$$-\beta(m_2k_{22}+2m_1k_{33}+m_{13}k_{26}+m_{14}k_{27}+m_5k_{28}+m_6k_{29}+m_1k_{32});$$

$$l_{24}=m_7-\beta m_{12};\qquad l_{25}=m_8-\beta m_{12};\qquad l_{26}=\beta m_{12}-\beta m_{11};$$

$$l_{27}=l_{23}-\beta m_{12}l_{22};\qquad l_{28}=\frac{e^{-m_{11}}}{e^{-m_{12}}}-1;\qquad l_{29}=\frac{l_{21}}{e^{-m_{12}}}+l_{22};$$

$$l_{30}=l_{24}-\frac{l_{26}}{l_{28}};\qquad l_{31}=l_{25}-\frac{l_{26}}{l_{28}};\qquad l_{32}=l_{27}-\frac{l_{26}l_{29}}{l_{28}};$$

$$m_1=\frac{1}{\alpha};\qquad m_2=\frac{\mathrm{Pr}}{\beta};\qquad m_3=\frac{1+\sqrt{1-4k_7}}{2};$$

$$m_4=\frac{1-\sqrt{1-4k_7}}{2};\qquad m_5=\frac{1+\sqrt{1-4\alpha k_7}}{2\alpha};\qquad m_6=\frac{1-\sqrt{1-4\alpha k_7}}{2\alpha};$$

$$m_7=\frac{\mathrm{Pr}+\sqrt{\mathrm{Pr}^2-4\mathrm{Pr}k_7}}{2};\qquad m_8=\frac{\mathrm{Pr}-\sqrt{\mathrm{Pr}^2-4\mathrm{Pr}k_7}}{2};\qquad m_9=m_3+1;$$

$$m_{10}=m_4+1;\qquad m_{11}=\frac{\mathrm{Pr}+\sqrt{\mathrm{Pr}^2-4\beta\mathrm{Pr}k_7}}{2\beta};$$

$$m_{12}=\frac{\mathrm{Pr}-\sqrt{\mathrm{Pr}^2-4\beta\mathrm{Pr}k_7}}{2\beta};\qquad m_{13}=m_1+m_5;\qquad m_{14}=m_1+m_6;$$

$$n_1 = J_1 + 1; \qquad n_2 = m_1 + J_2;$$

$$P_1 = e^{J_1}; \qquad P_2 = e^{J_1}; \qquad P_3 = \frac{A}{k_7}(C_2 e^1 + P);$$

$$P_4 = e^{-J_2}; \qquad P_5 = -e^{-J_2}; \qquad P_6 = k_8 e^{-m_1} + k_9;$$

$$P_7 = \frac{A}{k_7}(C_2 + P) - k_8 - k_9; \qquad P_8 = \frac{A}{k_7}C_2 - \alpha m_1 k_8; \qquad P_9 = J_1 - \alpha J_2;$$

$$P_{10} = P_8 - \alpha P_7 J_2; \qquad P_{11} = P_4 P_7 + P_6; \qquad P_{12} = P_9 + \frac{\alpha P_4}{P_5};$$

$$P_{13} = P_{10} + \frac{\alpha P_{11}}{P_5}; \qquad P_{14} = D_1 J_1 + D_2; \qquad P_{15} = D_2 J_1;$$

$$P_{16} = \frac{A C_2^2}{k_7}; \qquad P_{17} = \frac{A P C_2}{k_7}; \qquad P_{18} = -\frac{2 C_2 P_{14}}{F_2};$$

$$P_{19} = -\frac{2 C_2 P_{15}}{F_2}; \qquad P_{20} = \frac{2 C_2 P_{15} F_1}{F_2^2} + P_{18}; \qquad P_{21} = -\frac{2 P P_{14}}{F_4};$$

$$P_{22} = P_{21} + \frac{2 P P_{15} F_3}{F_4^2}; \qquad P_{23} = -\frac{2 P P_{15}}{F_4}; \qquad P_{24} = -\frac{2 P_{16}}{4 - 2\mathrm{Pr} + \mathrm{Pr} k_7};$$

$$P_{25} = -\frac{2 P_{17}}{1 - \mathrm{Pr} + \mathrm{Pr} k_7}; \qquad P_{26} = P_{24} + k_{11}; \qquad P_{27} = P_{25} + k_{12};$$

$$P_{28} = J_2 D_3 + D_4; \qquad P_{29} = J_2 D_4; \qquad P_{30} = -\frac{2 \alpha m_1 C_4 P_{28}}{F_6};$$

$$P_{31} = P_{30} + \frac{2 \alpha m_1 C_4 P_{28}}{F_6}; \qquad P_{32} = -\frac{2 \alpha m_1 C_4 P_{29}}{F_6}; \qquad P_{33} = -\frac{2 \alpha P P_{28}}{F_8};$$

$$P_{34} = P_{33} + \frac{2 \alpha P F_7 P_{29}}{F_8^2}; \qquad P_{35} = -\frac{2 \alpha P P_{29}}{F_8};$$

$$P_{36} = -\frac{2 \alpha m_1^2 C_4 k_8}{4 \beta m_1^2 - 2 m_1 \mathrm{Pr} + \mathrm{Pr} k_7}; \qquad P_{37} = -\frac{2 \alpha P m_1 k_8}{\beta m_1^2 - m_1 \mathrm{Pr} + \mathrm{Pr} k_7};$$

$$P_{38} = k_{23} + P_{36}; \qquad P_{39} = k_{24} + P_{37};$$

$$Q_1 = P_{20}e^{n_1} + P_{19}e^{n_1} + P_{23}e^{J_1} + P_{22}e^{J_1} + P_{26}e^2 + P_{27}e^1 + k_{10}e\mathrm{Pr} + k_{13};$$

$$Q_2 = P_{31}e^{-n_2} - P_{32}e^{-n_2} + P_{34}e^{-J_2} - P_{35}e^{-J_2} + P_{38}e^{-2m_1} + P_{39}e^{-m_1} + k_{22}e^{-m_2} + k_{25};$$

$$Q_3 = P_{20} + P_{23} + P_{26} + P_{27} + k_{10} + k_{13} - P_{31} - P_{34} - P_{38} - P_{39} - k_{22} - k_{25};$$

$$Q_4 = n_1P_{20} + P_{19} + J_1P_{23} + P_{22} + 2P_{26} + P_{27} + \mathrm{Pr}k_{10}$$
$$-\beta(n_2P_{31} + P_{32} + J_2P_{34} + P_{35} + 2m_1P_{38} + m_1P_{39} + m_2k_{22});$$

$$Q_5 = \beta J_4 + \beta; \qquad Q_6 = Q_4 - \frac{\beta Q_2}{e^{-J_2}}; \qquad Q_7 = J_3 - Q_5; \qquad Q_8 = Q_6 - Q_5Q_3;$$

$$R_1 = e^{n_1}\cos\delta_1; \qquad R_2 = e^{n_1}\sin\delta_1; \qquad R_3 = \frac{A}{k_7}(C_2e^1 + P);$$

$$R_4 = e^{-\gamma_2}\cos(D_2); \qquad R_5 = -e^{-\gamma_2}\sin(D_2); \qquad R_6 = k_8e^{-m_1} + k_9;$$

$$R_7 = \frac{A}{k_7}(C_2 + P) - k_8 - k_9; \qquad R_8 = \frac{A}{k_7}C_2 - \alpha m_1k_8; \qquad R_9 = \gamma_1 - \alpha\gamma_2;$$

$$R_{10} = R_8 - \alpha\gamma_2R_7; \qquad R_{11} = R_4R_7 + R_6; \qquad R_{12} = R_9 + \frac{\alpha\delta_2R_4}{R_5};$$

$$R_{13} = R_{10} + \frac{\alpha\delta_2R_{11}}{R_5}; \qquad R_{14} = e^{\gamma_3}\cos\delta_3; \qquad R_{16} = e^{\gamma_3}\sin\delta_3;$$

$$R_{17} = k_{10}e^{\mathrm{Pr}} + V_{12}e^2 + V_{13}e^1 + V_6e^{Z_1}(V_4\cos\delta_1 - V_5\sin\delta_1) + V_9e^{\gamma_1}(V_7\cos\delta_1 - V_8\sin\delta_1) + k_{13};$$

$$R_{18} = e^{-\gamma_4}\cos\delta_4; \qquad R_{19} = -e^{\gamma_4}\sin\delta_4;$$

$$R_{20} = k_{22}e^{-m_2} + V_{24}e^{2m_1} + V_{25}e^{-m_1} + V_{18}e^{-Z_2}(V_{16}\cos\delta_2 + V_{17}\sin\delta_2)$$
$$+V_{21}e^{-\gamma_2}(V_{19}\cos\delta_2 + V_{20}\sin\delta_2) + k_{25};$$

$$R_{21} = k_{10} + V_{12} + V_{13} + V_6V_4 + V_9V_7 + k_{13} - k_{22} - V_{24} - V_{25} - V_{18}V_{16} - V_{21}V_{19} - k_{25};$$

$$R_{22} = \mathrm{Pr}k_{10} + 2V_{12} + V_{13} + V_6(Z_1\gamma_4 - \delta_1V_5) + V_9(\gamma_1V_7 - \delta_1V_8)$$
$$-\beta(m_2k_{22} + 2m_1k_{24} + m_1V_{25} + V_{18}(Z_2V_{16} - \delta_2V_{17}) + V_{21}(\gamma_2V_{19} - \delta_2V_{20}));$$

$$R_{23} = \beta\gamma_4 - \frac{\beta\delta_4R_{18}}{R_{19}}; \quad R_{24} = R_{22} + \frac{\beta\delta_4R_{20}}{R_{19}}; \quad R_{25} = \gamma_3 - R_{23}; \qquad R_{26} = R_{24} - R_{23}R_{21};$$

$$S_1 = 2Z_1 - \mathrm{Pr}; \qquad S_2 = Z_1^2 - Z_1\mathrm{Pr} + \mathrm{Pr}k_7; \qquad S_3 = S_2 - \delta_1^2;$$

$$S_4 = 2\gamma_1 - \mathrm{Pr}; \qquad S_5 = \gamma_1^2 - \gamma_1\mathrm{Pr} + \mathrm{Pr}k_7; \qquad S_6 = S_5 - \delta_1^2;$$

$$S_7 = 2\beta Z_2 - \mathrm{Pr}; \qquad S_8 = \beta Z_2^2 - \mathrm{Pr}Z_2 + \mathrm{Pr}k_7; \qquad S_{10} = 2\beta\gamma_2 - \mathrm{Pr};$$

$$S_{11} = \beta\gamma_2^2 - \mathrm{Pr}\gamma_2 + \mathrm{Pr}k_7; \qquad S_{12} = S_{11} - \beta\delta_2^2;$$

$$V_1 = \gamma_1G_1 + \delta_1G_2; \qquad V_2 = \gamma_1G_2 - \delta_1G_1; \qquad V_3 = \frac{A}{k_7}C_2;$$

$$V_4 = S_1\delta_1V_2 - S_3V_1; \qquad V_5 = S_1\delta_1V_1 + S_3V_1; \qquad V_6 = \frac{2C_2}{\delta_1^2S_4^2 + S_3^2};$$

$$V_7 = S_4\delta_1V_1 - S_6V_1; \qquad V_8 = S_4\delta_1V_1 + S_6V_2; \qquad V_9 = \frac{2P}{\delta_1^2S_4^2 + S_6^2};$$

$$V_{10} = -\frac{2C_2V_3}{4 - 2\mathrm{Pr} + \mathrm{Pr}K_7}; \qquad V_{11} = -\frac{2PV_3}{1 - \mathrm{Pr} + \mathrm{Pr}K_7}; \qquad V_{12} = k_{11} + V_{10};$$

$$V_{13} = k_{12} + V_{11}; \qquad V_{14} = \gamma_2G_3 + \delta_2G_4; \qquad V_{15} = \gamma_2G_4 - \delta_2G_3;$$

$$V_{16} = \delta_2S_7V_{15} - S_9V_{14}; \qquad V_{17} = \delta_2S_7V_{14} + S_9V_{15}; \qquad V_{18} = \frac{2\alpha m_1C_4}{\delta_1^2S_7^2 + S_9^2};$$

$$V_{19} = \delta_2S_{10}V_{15} - \rho_{12}V_{14}; \qquad V_{20} = \delta_2S_{10}V_{14} + S_{12}V_{15}; \qquad V_{21} = \frac{2\alpha P}{\delta_2^2\rho_{10}^2 + \rho_{10}^2};$$

$$V_{22} = -\frac{2\alpha m_1^2 C_4 k_8}{4m_1^2\beta - 2m_1\beta\mathrm{Pr} + \mathrm{Pr}k_7}; \qquad V_{23} = -\frac{2\alpha m_1 P k_8}{m_1^2\beta - m_1\mathrm{Pr} + \mathrm{Pr}k_7};$$

$$V_{24} = k_{23} + V_{22}; \qquad V_{25} = k_{24} + V_{23};$$

$$Z_1 = \gamma_1 + 1; \qquad Z_2 = m_1 + \gamma_2;$$

$$\gamma_1 = \frac{1}{2}; \qquad \gamma_2 = \frac{1}{2}; \qquad \gamma_3 = \frac{\mathrm{Pr}}{2}; \qquad \gamma_4 = \frac{\mathrm{Pr}}{2\beta};$$

$$\delta_1 = \frac{\sqrt{4k_7 - 1}}{2}; \qquad \delta_2 = \frac{\sqrt{4\alpha k_7 - 1}}{2}; \qquad \delta_3 = \frac{\sqrt{4\mathrm{Pr}k_7 - \mathrm{Pr}^2}}{2}; \qquad \delta_4 = \frac{\sqrt{4\beta\mathrm{Pr}k_7 - \mathrm{Pr}^2}}{2\beta}.$$

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