OSCILLATORY HARTMANN TWO-FLUID FLOW AND HEAT TRANSFER IN A HORIZONTAL CHANNEL

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An unsteady Hartmann flow of two immiscible fluids through a horizontal channel with time-dependent oscillatory wall transpiration velocity is investigated. One of the fluids is assumed to be electrically conducting while the other fluid and the channel walls are assumed to be electrically insulating. Separate solutions for each fluid are obtained and these solutions are matched at the interface using suitable matching conditions. The partial differential equations governing the flow and heat transfer are transformed to ordinary differential equations and closed-form solutions are obtained in both fluids' regions of the channel for steady and unsteady conditions. The closed-form results are presented graphically for various values of the Hartmann number, frequency parameter, periodic frequency parameter viscosity and conductivity ratios as well as the Prandtl number to show their effect on the flow and heat transfer characteristics.

Key words: immiscible fluids, Hartmann flow, unsteady flow, channel flow, wall suction/injection.

1. Introduction

The unsteady laminar boundary-layer theory and flow response due to imposed oscillations has received much attention. Lighthil (1954) was the first to have studied the unsteady forced flow of a viscous incompressible fluid past a flat plate and a circular cylinder with small amplitude oscillation in free stream. The corresponding problem of an unsteady free convection flow along a vertical plate with oscillating surface temperature was studied by Nanda and Sharma (1963). Later, Muhuri and Maiti (1967) and Verma (1982) analyzed the effect of oscillation of the surface temperature on the unsteady free convection from a horizontal pipe. Hossain *et al*. (1998) used the linearized theory to study the free convection boundary-layer flow of electrically conducting fluid along a vertical plate due to surface temperature oscillations.

All these studies pertain to a single-fluid model. Most of the problems relating to petroleum industry, geophysics, plasma physics, magneto-fluid dynamics, etc., involve multi-fluid flow situations. The study of the interaction of the geomagnetic field with the fluid (i.e., hot springs) in the geothermal regions arises in geophysics. Once we know the interaction of the geomagnetic field with the flow we can easily determine, using the energy equation, the temperature distribution. The temperature is used to run the turbine across a magnetic field to generate electricity. Rudraiah *et al.* (1975) studied the Hartmann flow past a permeable bed in the presence of a transverse magnetic field with an interface at the surface of the permeable bed.

Both theoretical and experimental work is found in the literature on a stratified laminar flow of two immiscible fluids in a horizontal pipe. In the petroleum industry, as well as in other engineering fields, a stratified two-phase flow often occurs. There have been some experimental and analytical studies on hydrodynamic aspects of the two-phase flow reported in the recent literature. The interest in these types of

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problems stems from the possibility of reducing the power required to pump oil in a pipeline by suitable addition of water. The first investigations were associated with the LM-MFM generator project at the Argone National Laboratory. Packham and Shail (1971) analyzed a stratified laminar flow of two immiscible liquids in a horizontal pipe. The Hartmann flow of a conducting fluid in a channel with a layer of non-conducting fluid between the upper channel wall and the conducting fluid was studied by Shail (1973). He found that an increase of order *30%* could be achieved in the flow rate for suitable ratios of depths and viscosities of the two fluids. Loharsbi and Sahai (1987) dealt the with two-phase MHD flow and heat transfer in a parallel-plate channel. Both phases were incompressible and the flow was assumed to be steady, one-dimensional and fully developed. The study was expected to be useful in the understanding of the effect of the presence of slag layers on the heat transfer characteristics of a coal-fired MHD generator. Alireza and Sahai (1990) studied the effect of temperature-dependent transport properties on the developing MHD flow and heat transfer in a parallel plate channel whose walls were held at constant and equal temperatures. Following the work of Alireaz and Sahai (1990), Malashetty and Umavathi (1997) and Malashetty *et al.* (2001a, b) studied the two-phase MHD flow and heat transfer in an inclined channel. Chamkha (2000) reported analytical solutions for the flow of two-immiscible fluids in porous and non-porous parallel-plates channels.

The above investigations were carried out for steady flow situations. However, a significant number of practical problems dealing with immiscible fluids are unsteady. Chamkha (2004) studied the unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption. Keeping in view the wide range of applications in geophysics and MHD generators, an attempt is made to analyze the flow and heat transfer of the unsteady Hartmann flow of two immiscible fluids in a horizontal channel. One of the fluids is assumed to be electrically conducting, while the other fluid and the channel walls are assumed to be electrically insulating.

2. Mathematical formulation

The geometry under consideration consists of two infinite parallel plates extending in the X and Z directions (Fig.1). The regions $0 \le y \le h$ and $-h \le y \le 0$ are denoted as Region-I and Region-II, respectively. The fluid in Region-I is an electrically-conducting fluid having density ρ_1, viscosity μ_1 and thermal conductivity K_1. Region-II is filled with a electrically-non-conducting fluid having density ρ_2, viscosity μ_2 and thermal conductivity K_2. A constant magnetic field of strength B_0 is applied transverse to the flow direction. The magnetic Reynolds number is assumed to be small so that the induced magnetic field is neglected. Also, it is assumed that no electric field exists and that the Hall effect of magnetohydrodynamics is negligible. However, the effect of Joule heating (magnetic dissipation) of the electrically-conducting fluid is included in the model.

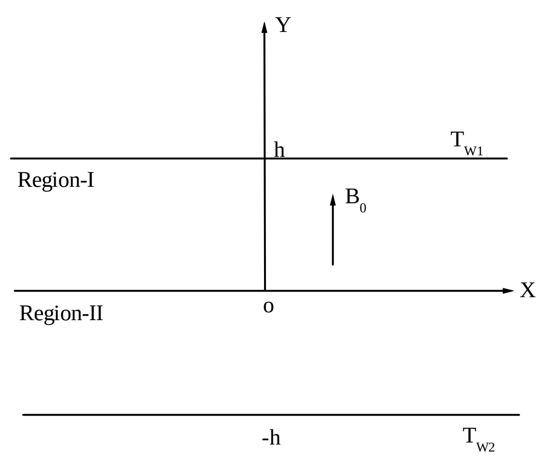

Fig.1. Physical configuration.

It is assumed that the flow is unsteady, fully developed and that all fluids properties are constants. The flow in both regions is assumed to be driven by a common pressure gradients $\left(-\frac{\partial p}{\partial x}\right)$ and temperature gradients $\Delta T = T_{w1} - T_{w2}$ where T_{w1} is the temperature of the boundary at $y = h$ and T_{w2} is the temperature of the boundary at $y = -h$.

Under these assumptions the governing equations of motion and energy (Loharsbi and Sahai, 1988) are:

Region-I

$$\frac{\partial v_1}{\partial y} = 0, \tag{2.1}$$

$$\rho_0\left(\frac{\partial u_1}{\partial t} + v_1\frac{\partial u_1}{\partial y}\right) = \mu_1\frac{\partial^2 u_1}{\partial y^2} - \frac{\partial p}{\partial x} - \sigma_1 u_1 B_0^2, \tag{2.2}$$

$$\rho_0 Cp\left(\frac{\partial T_1}{\partial t} + v_1\frac{\partial T_1}{\partial y}\right) = K_1\frac{\partial^2 T_1}{\partial y^2} + \mu_1\left(\frac{\partial u_1}{\partial y}\right)^2 + \sigma_1 B_0^2 u_1^2. \tag{2.3}$$

Region-II

$$\frac{\partial v_2}{\partial y} = 0, \tag{2.4}$$

$$\rho_0\left(\frac{\partial u_2}{\partial t} + v_2\frac{\partial u_2}{\partial y}\right) = \mu_2\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial p}{\partial x}, \tag{2.5}$$

$$\rho_0 Cp\left(\frac{\partial T_2}{\partial t} + v_2\frac{\partial T_2}{\partial y}\right) = K_2\frac{\partial^2 T_2}{\partial y^2} + \mu_2\left(\frac{\partial u_2}{\partial y}\right)^2 \tag{2.6}$$

where u is the x-component of fluid velocity, v is the y-component of fluid velocity and T is the fluid temperature. The subscripts *1* and *2* correspond to Region-I and Region-II, respectively. The boundary conditions on velocity are the no-slip boundary conditions which required that the x-component of velocity must vanish at the wall. The boundary conditions on temperature are isothermal conditions. We also assume the continuity of velocity, shear stress, temperature and heat flux at the interface between the two fluid layers at $y = 0$.

The hydrodynamic boundary and interface conditions for the two fluids can then be written as

$$\begin{aligned}
&u_1(h) = 0, \\
&u_2(-h) = 0, \\
&u_1(0) = u_2(0), \\
&\mu_1\frac{\partial u_1}{\partial y} = \mu_2\frac{\partial u_2}{\partial y} \quad \text{at} \quad y = 0.
\end{aligned} \tag{2.7}$$

The thermal boundary and interface conditions on temperature for both fluids are given by

$$T_1(h) = T_{w1}\,,$$

$$T_2(-h) = T_{w2}\,,$$

$$T_1(0) = T_2(0)\,, \tag{2.8}$$

$$K_1 \frac{\partial T_1}{\partial y} = K_2 \frac{\partial T_2}{\partial y} \quad \text{at} \quad y = 0\,.$$

The continuity equations of both fluids (Eqs (2.1) and (2.4)) imply that v_1 and v_2 are independent of y, they can be utmost a function of time alone. Hence, we can write (assuming $v_1 = v_2 = v$)

$$v = v_0\left(1 + \varepsilon A e^{i\omega t}\right) \tag{2.9}$$

where A is a real positive constant. ω is the frequency parameter and ε (a constant) is small such that $\varepsilon A \le 1$. Here, it is assumed that the transpiration velocity varies periodically with time about a non-zero constant mean v_0. When $\varepsilon A = 0$, the case of constant transpiration velocity is recovered. By use of the following non-dimensional quantities

$$u_i = \bar{u}_1 u_i^*, \quad y = hy^*, \quad t = \frac{h^2}{\nu} t^*, \quad v = \frac{\nu}{h} v^* = \frac{v}{|v_0|}, \quad \omega = \frac{\nu}{h^2}\omega^*,$$

$$P = \frac{h^2}{\mu_1 \bar{u}_1}\left(\frac{\partial P}{\partial x}\right), \quad \theta = \frac{T - T_w}{\bar{u}_1 \mu_1 / k_1}, \quad M = B_0 h \sqrt{\frac{\sigma_1}{\mu_1}}, \tag{2.10}$$

and for simplicity dropping the asterisks, Eqs (2.2), (2.3), (2.5) and (2.6) become
Region-I

$$\frac{\partial u_1}{\partial t} + v\frac{\partial u_1}{\partial y} = P + \frac{\partial^2 u_1}{\partial y^2} - M^2 u_1\,, \tag{2.11}$$

$$\frac{\partial \theta_1}{\partial t} + v\frac{\partial \theta_1}{\partial y} = \frac{1}{\Pr}\left(\frac{\partial^2 \theta_1}{\partial y^2} + \left(\frac{\partial u_1}{\partial y}\right)^2 + M^2 u_1^2\right). \tag{2.12}$$

Region-II

$$\frac{\partial u_2}{\partial t} + v\frac{\partial u_2}{\partial y} = P + \alpha \frac{\partial^2 u_2}{\partial y^2}\,, \tag{2.13}$$

$$\frac{\partial \theta_2}{\partial t} + v\frac{\partial \theta_2}{\partial y} = \frac{\beta}{\Pr}\frac{\partial^2 \theta_2}{\partial y^2} + \frac{\alpha}{Pr}\left(\frac{\partial u_2}{\partial y}\right)^2 \tag{2.14}$$

where $\Pr = \dfrac{\rho_0 \nu Cp}{K_1}$ is the Prandtl number, $\alpha = \dfrac{\mu_2}{\mu_1}$ is the ratio of viscosities and $\beta = \dfrac{K_2}{K_1}$ is the ratio of thermal conductivities.

The hydrodynamic and thermal boundary and interface conditions for both fluids in a non-dimensional form become

$$
\begin{aligned}
& u_1(1) = 0, \\
& u_2(-1) = 0, \\
& u_1(0) = u_2(0), \\
& \frac{\partial u_1}{\partial y} = \alpha \frac{\partial u_2}{\partial y} \quad \text{at} \quad y = 0,
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
& \theta_1(1) = 1, \\
& \theta_2(-1) = 0, \\
& \theta_1(0) = \theta_2(0), \\
& \frac{\partial \theta_1}{\partial y} = \beta \frac{\partial \theta_2}{\partial y} \quad \text{at} \quad y = 0.
\end{aligned}
\tag{2.16}
$$

3. Closed-form solutions

The governing momentum Eqs (2.11), (2.13) along with the energy Eqs (2.12) and (2.14) are solved subject to the boundary and interface conditions (2.15) and (2.16) for the velocity and temperature distributions in both regions. These equations are coupled partial differential equations that cannot be solved in a closed form. However, it can be reduced to ordinary differential equations by assuming

$$u_1(y,t) = u_{10}(y) + \varepsilon e^{i\omega t} u_{11}(y), \qquad u_2(y,t) = u_{20}(y) + \varepsilon e^{i\omega t} u_{21}(y), \tag{3.1}$$

$$\theta_1(y,t) = \theta_{10}(y) + \varepsilon e^{i\omega t} \theta_{11}(y), \qquad \theta_2(y,t) = \theta_{20}(y) + \varepsilon e^{i\omega t} \theta_{21}(y) \ . \tag{3.2}$$

This is a valid assumption because of the choice of v as defined in Eq.(2.9) that the amplitude $\varepsilon A << 1$.

By substituting Eqs (3.1) and (3.2) into Eqs (2.11) to (2.16), keeping the harmonic and non-harmonic terms and neglecting the higher-order terms of ε^2, one obtains the following pairs of equations:

Region-I

Non-periodic coefficients

$$\frac{d^2 u_{10}}{dy^2} - \frac{du_{10}}{dy} - M^2 u_{10} + P = 0, \tag{3.3}$$

$$\frac{d^2\theta_{10}}{dy^2} - \Pr\frac{d\theta_{10}}{dy} = -\left(M^2 u_{10}^2 + \left(\frac{du_{10}}{dy}\right)^2\right). \tag{3.4}$$

Periodic coefficients

$$\frac{d^2u_{11}}{dy^2} - \frac{du_{11}}{dy} - \left(M^2 + i\omega\right)u_{11} = A\frac{du_{10}}{dy}, \tag{3.5}$$

$$\frac{d^2\theta_{11}}{dy^2} - \Pr\frac{d\theta_{11}}{dy} - i\omega\Pr\theta_{11} = \Pr A\frac{d\theta_{10}}{dy} - 2M^2u_{10}u_{11} - 2\frac{du_{10}}{dy}\frac{du_{11}}{dy}. \tag{3.6}$$

Region -II
Non-periodic coefficients

$$\alpha\frac{d^2u_{20}}{dy^2} - \frac{du_{20}}{dy} + P = 0, \tag{3.7}$$

$$\beta\frac{d^2\theta_{20}}{dy^2} - \Pr\frac{d\theta_{20}}{dy} = -\alpha\left(\frac{du_{20}}{dy}\right)^2. \tag{3.8}$$

Periodic coefficients

$$\alpha\frac{d^2u_{21}}{dy^2} - \frac{du_{21}}{dy} - i\omega u_{21} = A\frac{du_{20}}{dy}, \tag{3.9}$$

$$\beta\frac{d^2\theta_{21}}{dy^2} - \Pr\frac{d\theta_{21}}{dy} - i\omega\Pr\theta_{21} = \Pr A\frac{d\theta_{20}}{dy} - 2\alpha\frac{du_{20}}{dy}\frac{du_{21}}{dy}. \tag{3.10}$$

The corresponding boundary conditions become as follows
Non-periodic coefficients

$$\begin{aligned} &u_{10}(1) = 0, \\ &u_{20}(-1) = 0, \\ &u_{10}(0) = u_{20}(0), \\ &\frac{\partial u_{10}}{\partial y} = \alpha\frac{\partial u_{20}}{\partial y} \quad \text{at} \quad y = 0. \end{aligned} \tag{3.11}$$

Periodic coefficients

$$u_{11}(1) = 0,$$

$$u_{21}(-1)=0,$$
$$u_{11}(0)=u_{21}(0),\tag{3.12}$$
$$\frac{\partial u_{11}}{\partial y}=\alpha\frac{\partial u_{21}}{\partial y}\quad \text{at}\quad y=0.$$

Non-periodic coefficients

$$\theta_{10}(1)=1,$$
$$\theta_{20}(-1)=0,$$
$$\theta_{10}(0)=\theta_{20}(0),\tag{3.13}$$
$$\frac{\partial\theta_{10}}{\partial y}=\beta\frac{\partial\theta_{20}}{\partial y}\quad \text{at}\quad y=0.$$

Periodic coefficients

$$\theta_{11}(1)=0,$$
$$\theta_{21}(-1)=0,$$
$$\theta_{11}(0)=\theta_{21}(0),\tag{3.14}$$
$$\frac{\partial\theta_{11}}{\partial y}=\beta\frac{\partial\theta_{21}}{\partial y}\quad \text{at}\quad y=0.$$

Equations (3.3) to (3.10) along with the boundary and interface conditions (3.11) to (3.14) represent a system of ordinary differential equations and conditions that can be solved in a closed form.

The solution of the non-periodic (harmonic) terms leads to the steady flow solutions for both fluids. Without going into detail, the steady-state velocity and temperature profiles can be shown to be

$$u_{10}=C_1e^{m_1y}+C_2e^{m_2y}+\frac{P}{M^2},\tag{3.15}$$

$$u_{20}=C_3+C_4e^{m_3y}+Py,\tag{3.16}$$

$$\theta_{10}=C_5+C_6e^{\mathrm{Pr}y}+K_4e^{2m_1y}+K_5e^{2m_2y}+K_6e^{m_4y}+K_7e^{m_1y}+K_8e^{m_2y}+K_9y,\tag{3.17}$$

$$\theta_{20}=C_7+C_8e^{m_5y}+K_{10}e^{2m_3y}+K_{11}e^{m_3y}+K_{12}y.\tag{3.18}$$

The solution of the periodic (non-harmonic) terms gives the transient velocity and temperature profiles in both regions of the channel (Region-I and Region-II). These solutions are given by

$$u_{11}=e^{A_1y}\left(XC_9\cos B_1y+XC_{10}\sin B_1y\right)+A_2e^{m_1y}+A_2e^{m_2y}+ \\ +i\left[e^{A_1y}\left(YC_9\cos B_1y+YC_{10}\sin B_1y\right)+B_2e^{m_1y}+B_2e^{m_2y}\right], \tag{3.19}$$

$$u_{21}=e^{A_2y}\left(XC_{11}\cos B_2y+XC_{10}\sin B_2y\right)+K_{13}e^{m_3y}+ \\ +i\left[e^{A_2y}\left(YC_{11}\cos B_2y+YC_{12}\sin B_2y\right)+K_{14}e^{m_3y}+K_{15}\right], \tag{3.20}$$

$$\begin{aligned}\theta_{11}&=e^{A_3y}\left(XC_{13}\cos B_3y+XC_{14}\sin B_3y\right)+P_{26}e^{2m_1y}+P_{27}e^{2m_2y}+P_{28}e^{m_4y}+\\&+P_{29}e^{m_1y}+P_{30}e^{m_2y}+e^{m_6y}\left(P_{31}\cos B_1y+P_{32}\sin B_1y\right)+\\&+e^{m_7y}\left(P_{33}\cos B_1y+P_{34}\sin B_1y\right)+e^{A_1y}\left(P_{35}\cos B_1+P_{36}\sin B_1y\right)+\\&+i\begin{bmatrix}e^{A_3y}\left(YC_{13}\cos B_3y+YC_{14}\sin B_3y\right)+Q_{26}e^{2m_1y}+Q_{27}e^{2m_2y}+Q_{28}e^{m_4y}+\\+Q_{29}e^{m_1y}+Q_{30}e^{m_2y}+e^{m_6y}\left(Q_{31}\cos B_1y+Q_{32}\sin B_1y\right)+\\+e^{m_7y}\left(Q_{33}\cos B_1y+Q_{34}\sin B_1y\right)+e^{A_1y}\left(Q_{35}\cos B_1+Q_{36}\sin B_1y\right)+K_{16}e^{\mathrm{Pr}y}+K_{17}\end{bmatrix},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\theta_{21}&=e^{A27y}\left(XC_{15}\cos B_{27}y+XC_{16}\sin B_{27}y\right)+A_{28}e^{m_5y}+K_{29}e^{2m_3y}+K_{30}e^{m_3y}+\\&+e^{m_8y}\left(P_{47}\cos B_4y+P_{48}\sin B_4y\right)+e^{A_4y}\left(P_{49}\cos B_4y+P_{50}\sin B_4y\right)+\\&+i\begin{bmatrix}e^{A27y}\left(YC_{15}\cos B_{27}y+YC_{16}\sin B_{27}y\right)+B_{28}e^{m_5y}+K_{31}e^{2m_3y}+K_{32}e^{m_3y}+\\+e^{m_8y}\left(Q_{47}\cos B_4y+Q_{48}\sin B_4y\right)+e^{A_4y}\left(Q_{49}\cos B_4y+Q_{50}\sin B_4y\right)+K_{26}\end{bmatrix}.\end{aligned} \tag{3.22}$$

The constants appearing in the above solutions are defined in the Appendix section.

4. Results and discussion

The problem of an unsteady two immiscible fluids flow through a horizontal channel is investigated analytically in the presence of an applied magnetic field transverse to the flow direction. The transpiration velocity is assumed to vary periodically with time about a non-zero constant mean velocity. The closed-form solutions reported for small ε, the coefficient of exponent of periodic frequency parameter in the previous section are evaluated for various parametric conditions. Although the correctness of the obtained results is not verified, the fact that the solutions satisfy all boundary and interface conditions (as will be seen in the graphical results) lend some confidence. The results are depicted graphically in Figs 2 to 11 to elucidate interesting features of the hydrodynamic and thermal state of the flow.

Figures 2 and 3 depict the effect of the Hartmann number M on the velocity and temperature fields, respectively. The application of a transverse magnetic field normal to the flow direction has a tendency to create a drag-like force called the Lorentz force. This force has a decreasing effect on the flow velocity. It is seen from Figs 2 and 3 that the magnetic field flattens out the velocity and temperature profiles and reduces the flow energy transfer as in the case of the steady-state Hartmann flow problem. Also, as expected, the influence of the Hartmann number on the velocity profiles is more pronounced in the channel region containing the electrically-conducting fluid compared to that containing the electrically non-conducting fluid. Furthermore, it is observed that the peak value in the velocity profile moves towards Region-II as the strength of the magnetic field increases.

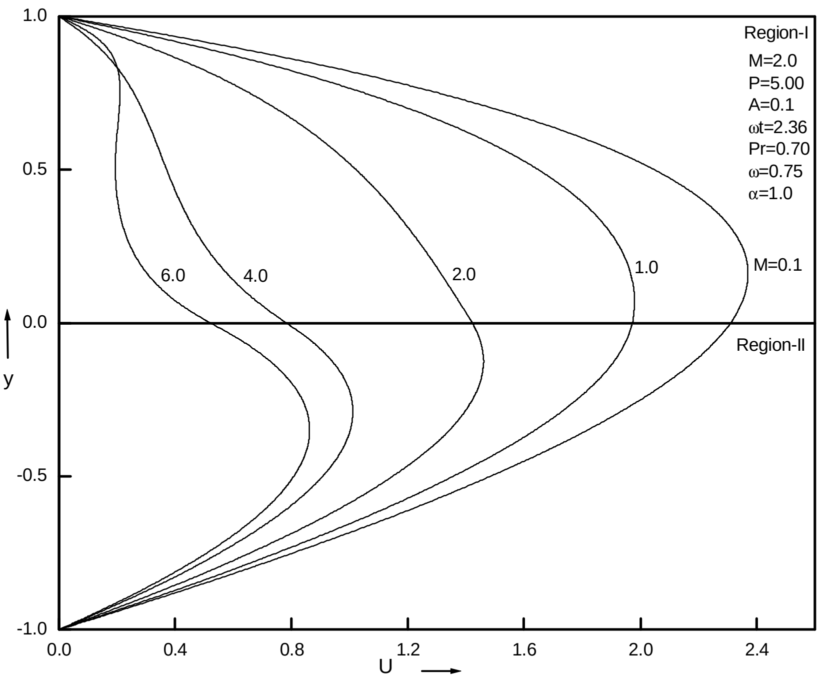

Fig.2. Velocity profiles for different values of the Hartmann number M.

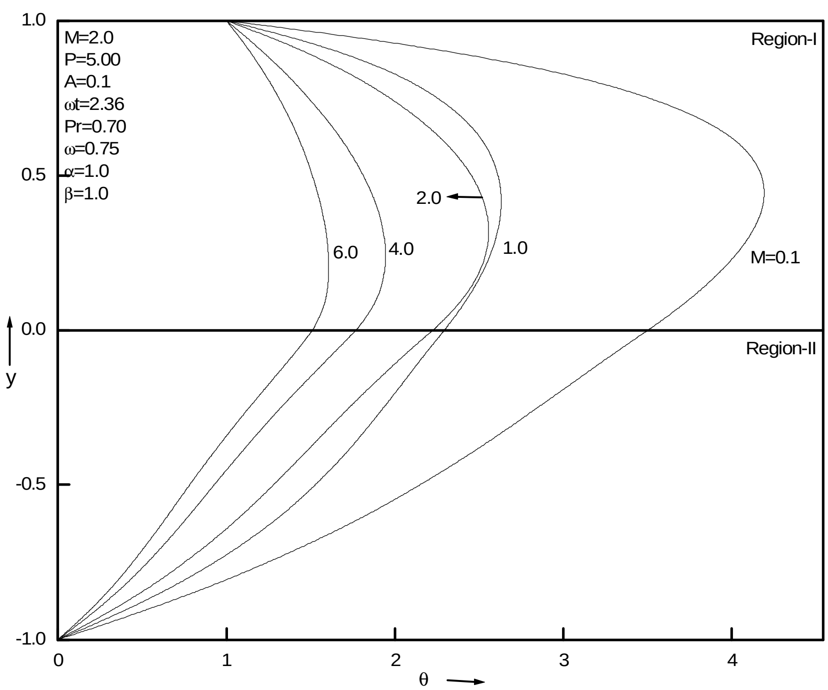

Fig.3. Temperature profiles for different values of the Hartmann number M.

The effect of the frequency parameter εA of the transpiration velocity on the velocity and temperature profiles is shown in Figs 4 and 5, respectively. In finding the closed-form solutions, it was assumed that ε is small and A is a real positive constant such that $\varepsilon A \leq 1$ and hence, in these figures the condition $\varepsilon A \leq 1$ is satisfied. It should be noted that in these figures the condition $\varepsilon A = 0$ corresponds to the case of constant transpiration velocity. It is predicted that as the frequency parameter increases, both the velocity and temperature profiles increase. This is expected since as εA increases, the amplitude of the periodic oscillations of the transpiration velocity increases resulting in an increased flow in the channel. Also, the maximum velocity in the channel tends to move above the channel centerline towards Region-I as the frequency parameter increases as is clear from Fig.4.

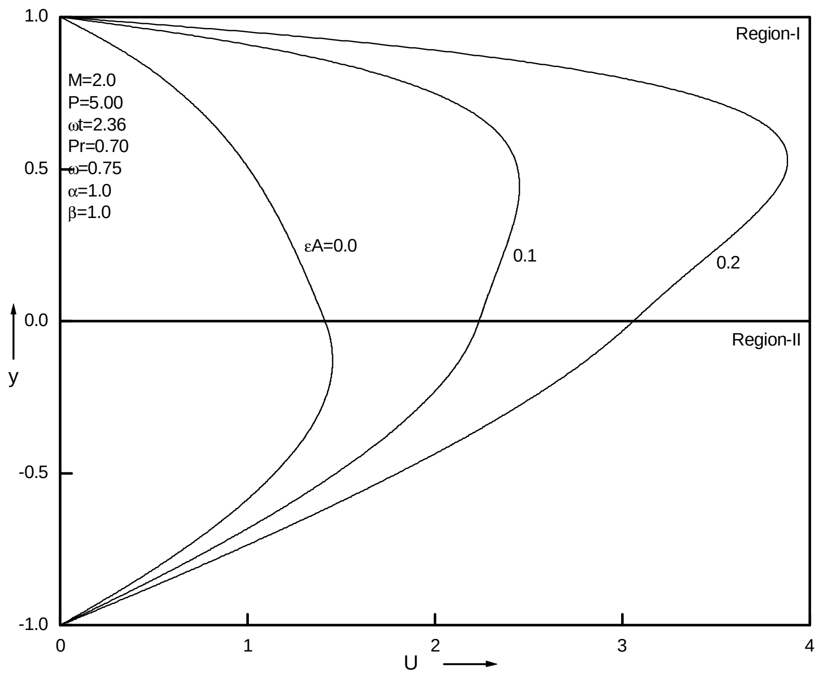

Fig.4. Velocity profiles for different value of the frequency parameter εA.

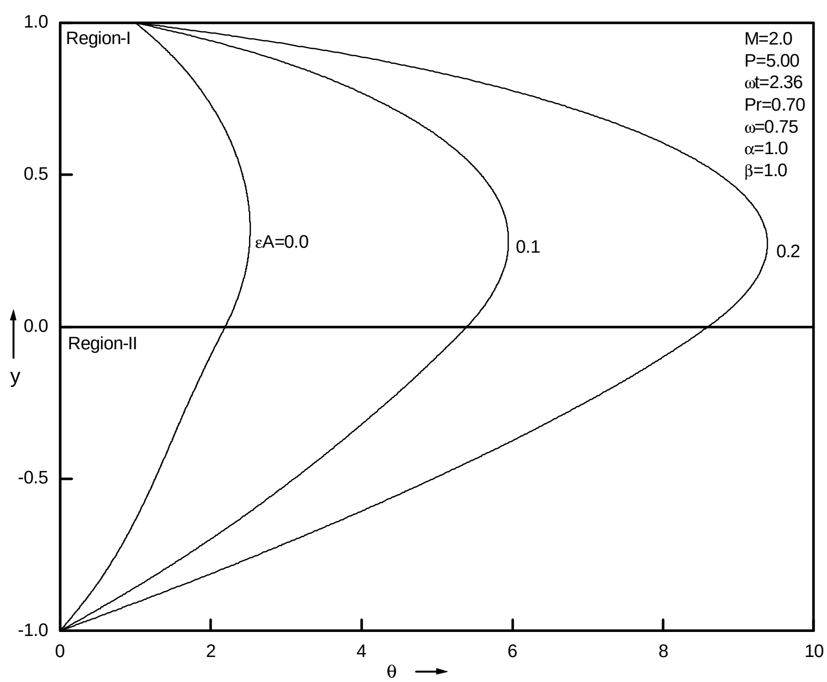

Fig.5. Temperature profiles for different values of the frequency parameter εA.

Figures 6 and 7 show representative velocity and temperature profiles for various values of the periodic frequency parameter ωt, respectively. Increasing the periodic frequency parameter has the effect of increasing the flow and heat transfer characteristics. This causes both the velocity and temperature fields in the channel to increase as evident from Figs 6 and 7. By a comparison with Figs 4 and 5, it is noted that the increase in both the velocity and temperature profiles as ωt increases is minimal when compared with the changes in these profiles as εA is increased.

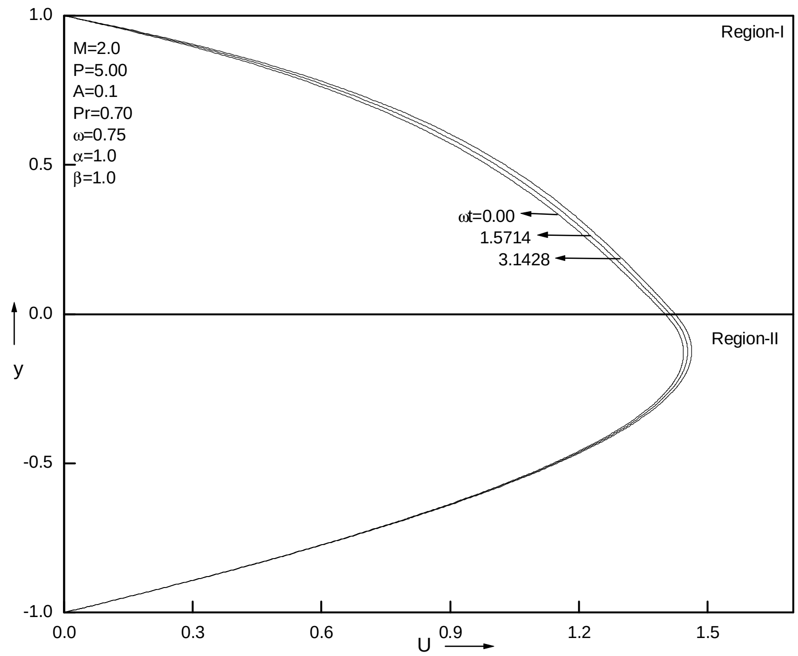

Fig.6. Velocity profiles for different values of the periodic frequency parameter ωt.

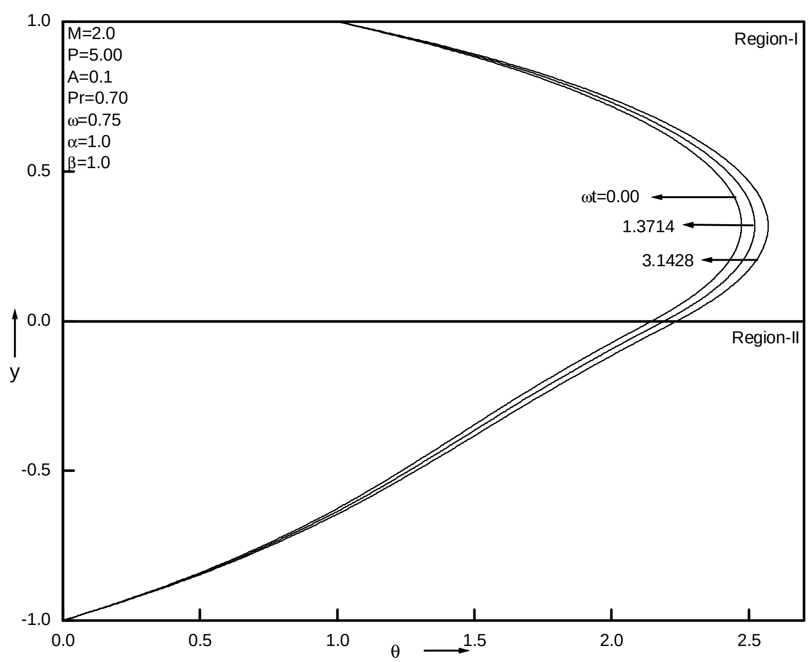

Fig.7. Temperature profiles for different values of the periodic frequency parameter ωt.

The effect of the viscosity ratio α on the velocity and temperature distributions is shown in Figs 8 and 9, respectively. As the viscosity ratio increases, both the velocity and temperature profiles decrease. This is due to the fact that as the viscous effects increase, the fluids in both regions become thicker and hence, the flow velocity in the channel is reduced causing the temperature distribution in the channel to reduce as well. This is evident from Figs 8 and 9.

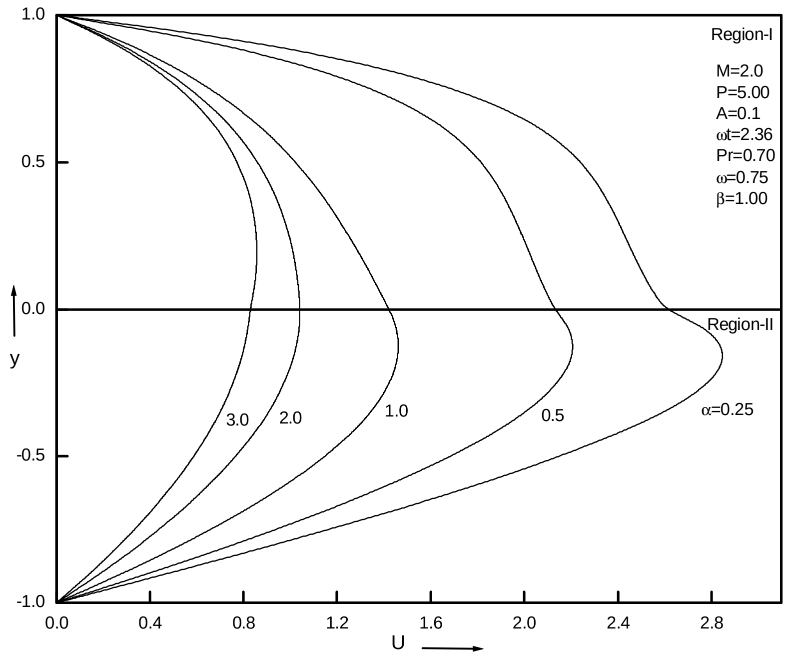

Fig.8. Velocity profiles for different values of the viscosity ratio α.

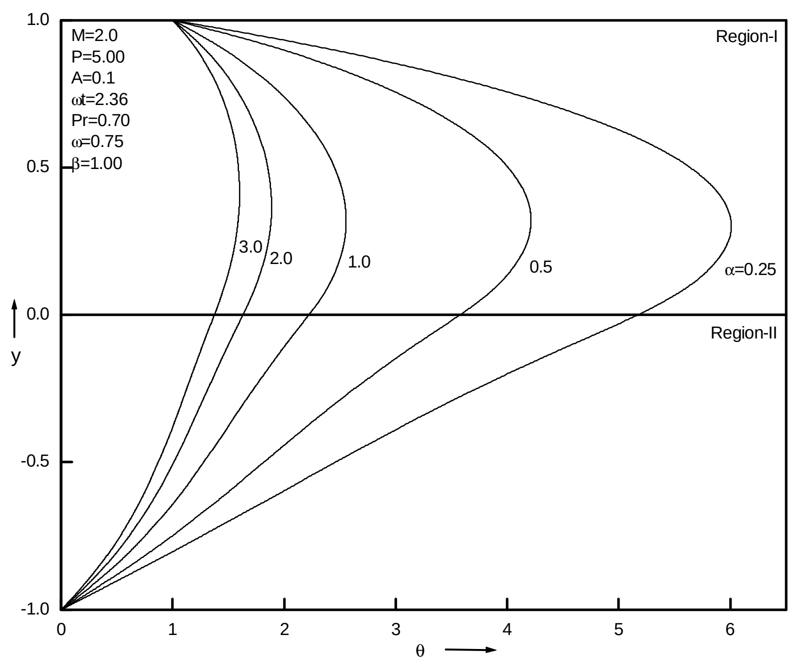

Fig.9. Temperature profiles for different values of the viscosity ratio α.

Figures 10 and 11 illustrate the influence of the conductivity ratio β and the Prandtl number Pr on the temperature profiles, respectively. As either of the conductivity ratio or the Prandtl number increases, the temperature field decreases. This thermal suppression is large in Region-II compared to Region-I due to the different boundary conditions on temperature. The Prandtl number which is the ratio of momentum diffusion to heat diffusion indicates that as the momentum diffusion increases, the heat diffusion effect will be reduced and hence the temperature decreases as the Prandtl number increases. Also, it is noted that the maximum temperature in the channel tends to move further above the channel centerline as either β or Pr increases and becomes very close to the upper wall of the channel for the case $\mathrm{Pr}=5$. These behaviors are clear from Figs 10 and 11.

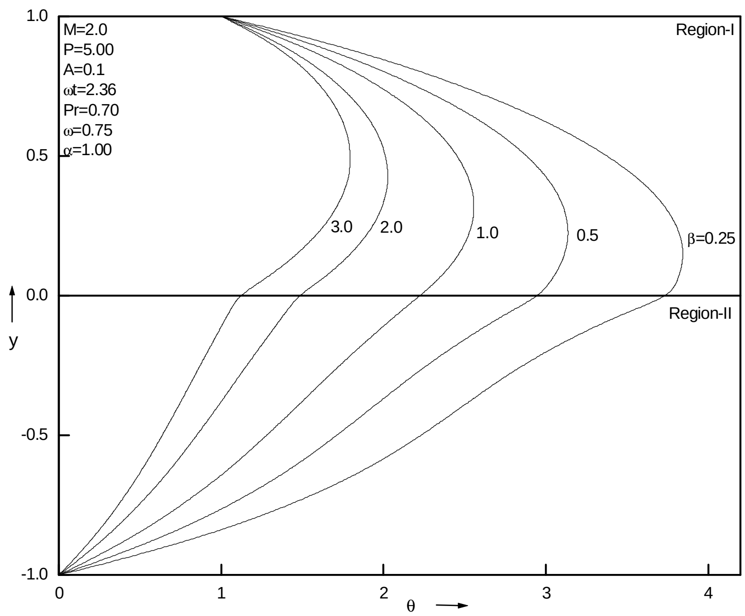

Fig.10. Temperature profiles for different values of the thermal conductivity ratio β.

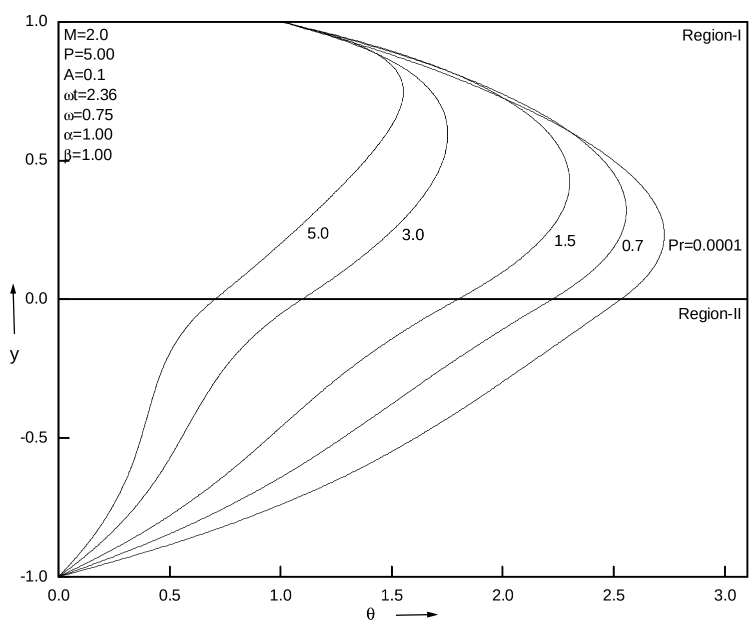

Fig.11. Temperature profiles for different values of the Prandtl number Pr.

5. Conclusion

The problem of an unsteady Hartmann flow of two immiscible fluids through a horizontal channel with time-dependent oscillatory wall transpiration velocity was investigated analytically. One of the fluids was assumed to be electrically conducting, while the other fluid and the channel walls were assumed to be electrically insulating. Separate closed-form solutions for each fluid were obtained taking into consideration suitable interface matching conditions. The closed-form results were numerically evaluated and presented graphically for various values of the Hartmann number, frequency parameter, periodic frequency parameter viscosity and conductivity ratios as well as the Prandtl number. It was found that the flow and heat transfer characteristics can be effectively controlled by the properties of the two fluids as well as the fluid suction/injection amount at the boundary.

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Nomenclature

A – real positive constant
C_P – specific heat at constant pressure
$\boldsymbol{g}$ – gravitational acceleration
K – thermal conductivity
M – Hartmann number
P – pressure
Pr – Prandtl number
T – temperature
T_w – wall temperature
t – time
u,v – velocity components of velocity along and perpendicular to the plates, respectively
$\bar{u}_1$ – average velocity
v_0 – scale of suction
ε – coefficient of periodic parameter
θ – non-dimensional temperature
μ – viscosity of fluid
ν – kinematic viscosity
ρ – fluid density
ω – frequency parameter
ωt – periodic frequency parameter

Subscripts

1, 2 – quantities for Region-I and Region-II, respectively

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Appendix

$$A_1 = \frac{1+\sqrt{r_1}\cos\left(\theta_1/2\right)}{2}, \qquad A_2 = \frac{Am_1C_1\left(m_1^2 - m_1 - m^2\right)}{\left(m_1^2 - m_1 - m^2\right)^2 + \omega^2},$$

$$A_3 = \frac{Am_2C_2\left(m_2^2 - m_2 - m^2\right)}{\left(m_2^2 - m_2 - m^2\right)^2 + \omega^2}, \qquad A_4 = \frac{1+\sqrt{r_2}\cos\left(\theta_2/2\right)}{2\alpha},$$

$$A_5 = \frac{\Pr + \sqrt{r_3}\cos\theta_3/2}{2}, \qquad A_6 = \frac{2m_1\Pr AK_4\left(4m_1^2 - 2m_1\Pr\right)}{\left(4m_1^2 - 2m_1\Pr\right)^2 + \omega^2\Pr^2},$$

$$A_7 = \frac{2m_2\Pr AK_5\left(4m_2^2 - 2m_2\Pr\right)}{\left(4m_2^2 - 2m_2\Pr\right)^2 + \omega^2\Pr^2}, \qquad A_8 = \frac{\Pr Am_4K_6\left(m_4^2 - m_4\Pr\right)}{\left(m_4^2 - m_4\Pr\right)^2 + \omega^2\Pr^2},$$

$$A_9 = \frac{\Pr A m_1 K_7 \left(m_1^2 - m_1 \Pr\right)}{\left(m_1^2 - m_1 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{10} = \frac{\Pr A m_2 K_8 \left(m_2^2 - m_2 \Pr\right)}{\left(m_2^2 - m_2 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{11} = \frac{-2M^2 C_1 A_2 \left(4m_1^2 - 2m_1 \Pr\right)}{\left(4m_1^2 - 2m_1 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{12} = \frac{-2M^2 C_2 A_3 \left(4m_2^2 - 2m_2 \Pr\right)}{\left(4m_2^2 - 2m_2 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{13} = \frac{-2M^2 P_1 \left(m_4^2 - m_4 \Pr\right)}{\left(m_4^2 - m_4 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{14} = \frac{-2M^2 P_2 \left(m_1^2 - m_1 \Pr\right)}{\left(m_1^2 - m_1 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{15} = \frac{-2M^2 P_3 \left(m_2^2 - m_2 \Pr\right)}{\left(m_2^2 - m_2 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{16} = \frac{-2M^2 C_1 B_2 \left(4m_1^2 - 2m_1 \Pr\right)}{\left(4m_1^2 - 2m_1 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{17} = \frac{-2M^2 C_2 B_3 \left(4m_2^2 - 2m_2 \Pr\right)}{\left(4m_2^2 - 2m_2 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{18} = \frac{-2M^2 Q_1 \left(m_4^2 - m_4 \Pr\right)}{\left(m_4^2 - m_4 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{19} = \frac{-2M^2 Q_2 \left(m_1^2 - m_1 \Pr\right)}{\left(m_1^2 - m_1 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{20} = \frac{-2M^2 Q_3 \left(m_2^2 - m_2 \Pr\right)}{\left(m_2^2 - m_2 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{21} = \frac{-2m_1^2 C_1 A_2 \left(4m_1^2 - 2m_1 \Pr\right)}{\left(4m_1^2 - 2m_1 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{22} = \frac{-2m_2^2 C_2 A_3 \left(4m_2^2 - 2m_2 \Pr\right)}{\left(4m_2^2 - 2m_2 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{23} = \frac{-2K_{22} \left(m_4^2 - m_4 \Pr\right)}{\left(m_4^2 - m_4 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{24} = \frac{-2m_1^2 C_1 B_2 \left(4m_1^2 - 2m_1 \Pr\right)}{\left(4m_1^2 - 2m_1 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{25} = \frac{-2m_2^2 C_2 B_3 \left(4m_2^2 - 2m_2 \Pr\right)}{\left(4m_2^2 - 2m_2 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{26} = \frac{-2K_{23} \left(m_4^2 - m_4 \Pr\right)}{\left(m_4^2 - m_4 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{27} = \frac{\Pr + \sqrt{r_3} \cos\left(\theta_3 / 2\right)}{2\beta}, \qquad A_{28} = \frac{\Pr A m_5 C_8 \left(\beta m_5^2 - m_5 \Pr\right)}{\left(\beta m_5^2 - m_5 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{29} = \frac{2\Pr A m_3 K_{10} \left(4\beta m_3^2 - 2m_3 \Pr\right)}{\left(4\beta m_3^2 - 2m_3 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{30} = \frac{\Pr A m_3 K_{11} \left(\beta m_3^2 - m_3 \Pr\right)}{\left(\beta m_3^2 - m_3 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{31} = \frac{-2\alpha m_3^2 C_4 K_{13} \left(4\beta m_3^2 - 2m_3 \Pr\right)}{\left(4\beta m_3^2 - 2m_3 \Pr\right)^2 + \omega^2 \Pr^2}, \qquad A_{32} = \frac{-2\alpha P m_3 K_{13} \left(\beta m_3^2 - m_3 \Pr\right)}{\left(\beta m_3^2 - m_3 \Pr\right)^2 + \omega^2 \Pr^2},$$

$$A_{33}=\frac{-2\alpha m_3^2C_4K_{14}\left(4\beta m_3^2-2m_3\mathrm{Pr}\right)}{\left(4\beta m_3^2-2m_3\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\quad A_{34}=\frac{-2\alpha Pm_3K_{14}\left(\beta m_3^2-m_3\mathrm{Pr}\right)}{\left(\beta m_3^2-m_3\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_1=\frac{\sqrt{r_1}\sin\left(\theta_1/2\right)}{2},\qquad B_2=\frac{Am_1C_1\omega}{\left(m_1^2-m_1-m^2\right)^2+\omega^2},$$

$$B_3=\frac{Am_2C_2\omega}{\left(m_2^2-m_2-m^2\right)^2+\omega^2},\qquad B_4=\frac{1+\sqrt{r_2}\sin\theta_2/2}{2\alpha},$$

$$B_5=\frac{\sqrt{r_3}\sin\left(\theta_3/2\right)}{2},\qquad B_6=\frac{2m_1\mathrm{Pr}^2\,AK_4\omega}{\left(4m_1^2-2m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_7=\frac{2m_2\mathrm{Pr}^2\,AK_5\omega}{\left(4m_2^2-2m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_8=\frac{\mathrm{Pr}^2\,Am_4K_6\omega}{\left(m_4^2-m_4\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_9=\frac{\mathrm{Pr}^2\,Am_1K_7\omega}{\left(m_1^2-m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{10}=\frac{\mathrm{Pr}^2\,Am_2K_8\omega}{\left(m_2^2-m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{11}=\frac{-2M^2C_1A_2\omega\mathrm{Pr}}{\left(4m_1^2-2m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{12}=\frac{-2M^2C_2A_3\omega\mathrm{Pr}}{\left(4m_2^2-2m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{13}=\frac{-2M^2P_1\omega\mathrm{Pr}}{\left(m_4^2-m_4\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{14}=\frac{-2M^2P_2\omega\mathrm{Pr}}{\left(m_1^2-m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{15}=\frac{-2M^2P_3\omega\mathrm{Pr}}{\left(m_2^2-m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{16}=\frac{-2M^2C_1B_2\omega\mathrm{Pr}}{\left(4m_1^2-2m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{17}=\frac{-2M^2C_2B_3\omega\mathrm{Pr}}{\left(4m_2^2-2m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{18}=\frac{-2M^2Q_1\omega\mathrm{Pr}}{\left(m_4^2-m_4\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{19}=\frac{-2M^2Q_2\omega\mathrm{Pr}}{\left(m_1^2-m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{20}=\frac{-2M^2Q_3\omega\mathrm{Pr}}{\left(m_2^2-m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{21}=\frac{-2m_1^2C_1A_2\omega\mathrm{Pr}}{\left(4m_1^2-2m_1\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},\qquad B_{22}=\frac{-2m_2^2C_2A_3\omega\mathrm{Pr}}{\left(4m_2^2-2m_2\mathrm{Pr}\right)^2+\omega^2\mathrm{Pr}^2},$$

$$B_{23}=\frac{-2K_{22}\omega \Pr}{\left(m_4^2-m_4\Pr\right)^2+\omega^2\Pr^2},\qquad B_{24}=\frac{-2m_1^2C_1B_2\omega\Pr}{\left(4m_1^2-2m_1\Pr\right)^2+\omega^2\Pr^2},$$

$$B_{25}=\frac{-2m_2^2C_2B_3\omega\Pr}{\left(4m_2^2-2m_2\Pr\right)^2+\omega^2\Pr^2},\qquad B_{26}=\frac{-2K_{23}\omega\Pr}{\left(m_4^2-m_4\Pr\right)^2+\omega^2\Pr^2},$$

$$B_{27}=\frac{\sqrt{r_3}\sin\left(\theta_3/2\right)}{2\beta},\qquad B_{28}=\frac{\Pr A m_5C_8\omega\Pr}{\left(\beta m_5^2-m_5\Pr\right)^2+\omega^2\Pr^2},$$

$$B_{29}=\frac{2\Pr A m_3K_{10}\omega\Pr}{\left(4\beta m_3^2-2m_3\Pr\right)^2+\omega^2\Pr^2},\qquad B_{30}=\frac{\Pr A m_3K_{11}\omega\Pr}{\left(\beta m_3^2-m_3\Pr\right)^2+\omega^2\Pr^2},$$

$$B_{31}=\frac{-2\alpha m_3^2C_4K_{13}\omega\Pr}{\left(4\beta m_3^2-2m_3\Pr\right)^2+\omega^2\Pr^2},\qquad B_{32}=\frac{-2\alpha P m_3K_{13}\omega\Pr}{\left(\beta m_3^2-m_3\Pr\right)^2+\omega^2\Pr^2},$$

$$B_{33}=\frac{-2\alpha m_3^2C_4K_{14}\omega\Pr}{\left(4\beta m_3^2-2m_3\Pr\right)^2+\omega^2\Pr^2},\qquad B_{34}=\frac{-2\alpha P m_3K_{14}\omega\Pr}{\left(\beta m_3^2-m_3\Pr\right)^2+\omega^2\Pr^2},$$

$$C_1=-\frac{\left(C_2e^{m2}m^2+P\right)}{m^2e^{m1}},\qquad C_2=\frac{m^2l_3e^{m1}-l_1P}{m^2\left(l_1e^{m2}-l_2e^{m1}\right)},\qquad C_3=C_1+C_2-C_4+\frac{P}{m^2},$$

$$C_4=m_1C_1+m_2C_2-\alpha P,\qquad C_5=-\left(C_6e^{\Pr}+l_4\right),\qquad C_6=\frac{l_8-l_4}{e^{Pr}-l_9},$$

$$C_7=C_5+C_6-C_8+l_6,\qquad C_8=\frac{\Pr C_6+l_7}{\beta m_5},$$

$$E_1=2m_6-\Pr,\qquad E_2=m_6^2-m_6\Pr-B_1^2,\qquad E_3=E_2^2+\omega^2\Pr^2-E_1^2B_1^2,$$

$$E_4=2m_7-\Pr,\qquad E_5=m_7^2-m_7\Pr-B_1^2,\qquad E_6=E_5^2+\omega^2\Pr^2-E_4^2B_1^2,$$

$$E_7=2A_1-\Pr,\qquad E_8=A_1^2-A_1\Pr-B_1^2,\qquad E_9=E_8^2+\omega^2\Pr^2-E_9^2B_1^2,$$

$$E_{10}=2\beta m_8-\Pr,\qquad E_{11}=\beta m_8^2-m_8\Pr-\beta B_4^2,\qquad E_{12}=E_{11}^2+\omega^2\Pr^2-E_{10}^2B_4^2,$$

$$E_{13}=2\beta A_4-\Pr,\qquad E_{14}=\beta A_4^2-A_4\Pr-\beta B_4^2,\qquad E_{15}=E_{14}^2+\omega^2\Pr^2-E_{13}^2B_4^2,$$

$$F_1=2E_1^2E_2,\qquad F_2=2E_1E_2^2-E_1E_3,\qquad F_3=-E_2E_3,$$

$F_4 = 2\omega \Pr E_1 E_2$, $F_5 = -\omega \Pr E_3$, $F_6 = 2E_4^2 E_5$,

$F_7 = 2E_4 E_5^2 - E_4 E_6$, $F_8 = -E_5 E_6$, $F_9 = 2\omega \Pr E_4 E_5$,

$F_{10} = -\omega \Pr E_6$, $F_{11} = 2E_7^2 E_8$, $F_{12} = 2E_7 E_8^2 - E_7 E_9$,

$F_{13} = -E_8 E_9$, $F_{14} = 2\omega \Pr E_7 E_8$, $F_{15} = -\omega \Pr E_9$,

$F_{16} = 2E_{10}^2 E_{11}$, $F_{17} = 2E_{10} E_{11}^2 - E_{10} E_{12}$, $F_{18} = -E_{11} E_{12}$,

$F_{19} = 2\omega \Pr E_{10} E_{11}$, $F_{20} = -\omega \Pr E_{12}$, $F_{21} = 2E_{13}^2 E_{14}$,

$F_{22} = 2E_{13} E_{14}^2 - E_{13} E_{15}$, $F_{23} = -E_{14} E_{15}$, $F_{24} = 2\omega \Pr E_{13} E_{14}$,

$F_{25} = -\omega \Pr E_{15}$,

$K_1 = C_1^2\left(m^2 + m_1^2\right)$, $K_2 = C_2^2\left(m^2 + m_2^2\right)$, $K_3 = 2C_1 C_2\left(m^2 + m_1 m_2\right)$,

$$K_4 = \frac{-K_1}{4m_1^2 - 2m_1 \Pr}, \quad K_5 = \frac{-K_2}{4m_2^2 - 2m_2 \Pr}, \quad K_6 = \frac{-K_3}{m_4^2 - m_4 \Pr},$$

$$K_7 = \frac{-2C_1 P}{m_1^2 - m_1 \Pr}, \quad K_8 = \frac{-2C_2 P}{m_2^2 - m_2 \Pr}, \quad K_9 = \frac{P^2}{m^2 \Pr},$$

$$K_{10} = \frac{-\alpha m_3^2 C_4^2}{4\beta m_3^2 - 2m_3 \Pr}, \quad K_{11} = \frac{-2\alpha m_3 C_4 P}{\beta m_3^2 - m_3 \Pr}, \quad K_{12} = \frac{\alpha P^2}{\Pr},$$

$$K_{13} = \frac{A m_3 C_4 \left(\alpha m_3^2 - m_3\right)}{\left(\alpha m_3^2 - m_3\right)^2 + \omega^2}, \quad K_{14} = \frac{A m_3 C_4 \omega}{\left(\alpha m_3^2 - m_3\right)^2 + \omega^2}, \quad K_{15} = \frac{AP}{\omega},$$

$$K_{16} = \frac{A \Pr C_6}{\omega}, \quad K_{17} = \frac{A K_9}{\omega}, \quad K_{19} = \frac{2M^2 C_1}{4E_1^2 E_2^2 B_1^2 + E_3^2},$$

$$K_{20} = \frac{2M^2 C_2}{4E_4^2 E_5^2 B_1^2 + E_6^2}, \quad K_{21} = \frac{2P}{4E_7^2 E_8^2 B_1^2 + E_9^2}, \quad K_{22} = m_1 m_2 (C_1 A_3 + C_2 A_2),$$

$$K_{23} = m_1 m_2 (C_1 B_3 + C_2 B_2), \quad K_{24} = \frac{2m_1 C_1}{4E_1^2 E_2^2 B_1^2 + E_3^2}, \quad K_{25} = \frac{2m_2 C_2}{4E_4^2 E_5^2 B_1^2 + E_6^2},$$

$$K_{26} = \frac{AK_{12}}{\omega}, \qquad K_{27} = \frac{2\alpha m_3 C_4}{4E_{10}^2 E_{11}^2 B_4^2 + E_{12}^2}, \qquad K_{28} = \frac{2\alpha P}{4E_{13}^2 E_{14}^2 B_4^2 + E_{15}^2},$$

$$l_1 = 1 + m_1\left(e^{-m3} - 1\right), \quad l_2 = 1 + m_1\left(e^{-m3} - 1\right), \qquad l_3 = \frac{P}{m^2} - p - \alpha\, p\left(e^{-m3} - 1\right),$$

$$l_4 = K_4 e^{2m_1} + K_5 e^{2m_2} + K_6 e^{m_4} + K_7 e^{m_1} + K_8 e^{m_2} + K_9 - 1,$$

$$l_5 = K_{10} e^{-2m_3} + K_{11}\, e^{-m_3} - K_{12}, \qquad l_6 = K_4 + K_5 + K_6 + K_7 + K_8 - K_{10} - K_{11},$$

$$l_7 = 2m_1 K_4 + 2m_2 K_5 + m_4 K_6 + m_1 K_7 + m_2 K_8 + K_9 - \beta\left(2m_3 K_{10} + m_3 K_{11} + K_{12}\right),$$

$$l_8 = l_6 + l_5 + \frac{\left(e^{-m5} - 1\right)}{\beta m_5} l_7, \qquad l_9 = 1 + \frac{\left(e^{-m_5} - 1\right)\Pr}{\beta m_5}, \quad l_{10} = e^{A1} \cos B_1,$$

$$l_{11} = e^{A1} \sin B_1, \qquad l_{12} = A_2 e^{m1} + A_3 e^{m2}, \qquad l_{13} = B_2 e^{m1} + B_3 e^{m2},$$

$$l_{14} = e^{-A4} \cos B_4, \qquad l_{15} = -e^{-A4} \sin B_4, \qquad l_{16} = K_{13} e^{-m3},$$

$$l_{17} = K_{14} e^{-m3} + K_{15}, \quad l_{18} = A_2 + A_3 - K_{13}, \qquad l_{19} = B_2 + B_3 - K_{14} - K_{15},$$

$$l_{20} = m_1 A_2 + m_2 A_3 - \alpha m_3 K_{13}, \qquad l_{21} = m_1 B_2 + m_2 B_3 - \alpha m_3 K_{14},$$

$$l_{22} = \frac{\alpha B_4 l_{14}}{l_{15}} - \alpha A_4, \quad l_{23} = \frac{\alpha B_4 l_{16}}{l_{15}} + l_{20}, \qquad l_{24} = \frac{\alpha B_4 l_{17}}{l_{15}} + l_{21},$$

$$l_{25} = A_1 + l_{22}, \qquad l_{26} = l_{18} l_{22} + l_{23}, \qquad l_{27} = l_{19} l_{22} + l_{24},$$

$$l_{28} = e^{A_5} \cos B_5, \qquad l_{29} = e^{A_5} \sin B_5,$$

$$l_{30} = P_{26} e^{2m_1} + P_{27} e^{2m_2} + P_{28} e^{m_4} + P_{29} e^{m_1} + P_{30} e^{m_2} + e^{m_6}\left(P_{31}\cos B_1 + P_{32}\sin B_1\right) + \\ + e^{m_7}\left(P_{33}\cos B_1 + P_{34}\sin B_1\right) + e^{A_1}\left(P_{35}\cos B_1 + P_{36}\sin B_1\right),$$

$$l_{31} = Q_{26} e^{2m_1} + Q_{27} e^{2m_2} + Q_{28} e^{m_4} + Q_{29} e^{m_1} + Q_{30} e^{m_2} + e^{m_6}\left(Q_{31}\cos B_1 + Q_{32}\sin B_1\right) + \\ + e^{m_7}\left(Q_{33}\cos B_1 + Q_{34}\sin B_1\right) + e^{A_1}\left(Q_{35}\cos B_1 + Q_{36}\sin B_1\right) + K_{16} e^{Pr} + K_{17},$$

$$l_{32} = e^{-A_{27}} \cos B_{27}, \qquad l_{33} = -e^{-A_{27}} \sin B_{27},$$

$$l_{34} = A_{28} e^{-m_5} + K_{29} e^{-2m_3} + K_{30} e^{-m_3} + e^{-m_8}\left(P_{47}\cos B_4 - P_{48}\sin B_4\right) + \\ + e^{A_4}\left(P_{49}\cos B_4 - P_{50}\sin B_4\right),$$

$$l_{35} = B_{28}e^{-m_5} + K_{31}e^{-2m_3} + K_{32}e^{-m_3} + e^{-m_8}\left(Q_{47}\cos B_4 - Q_{48}\sin B_4\right) + \\ + e^{A_4}\left(Q_{49}\cos B_4 - Q_{50}\sin B_4\right) + K_{26},$$

$$l_{36} = P_{26} + P_{27} + P_{28} + P_{29} + P_{30} + P_{31} + P_{33} + P_{35} - A_{28} - K_{29} - K_{30} - P_{47} - P_{49},$$

$$l_{37} = Q_{26} + Q_{27} + Q_{28} + Q_{29} + Q_{30} + Q_{31} + Q_{33} + Q_{35} + K_{16} + \\ + K_{17} - B_{28} - K_{31} - K_{32} - Q_{47} - Q_{49} - K_{26},$$

$$l_{38} = 2m_1P_{26} + 2m_2P_{27} + m_4P_{28} + m_1P_{29} + m_2P_{30} + m_6P_{31} + \\ + B_1P_{32} + m_7P_{33} + B_1P_{34} + A_1P_{35} + B_1P_{36} + \\ - \beta\left(m_5A_{28} + 2m_3K_{29} + m_3K_{30} + m_8P_{47} + B_4P_{48} + A_4P_{49} + B_4P_{50}\right),$$

$$l_{39} = 2m_1Q_{26} + 2m_2Q_{27} + m_4Q_{28} + m_1Q_{29} + m_2Q_{30} + m_6Q_{31} + \\ + B_1Q_{32} + m_7Q_{33} + B_1Q_{34} + A_1Q_{35} + B_1Q_{36} + \\ - \beta\left(m_5B_{28} + 2m_3K_{31} + m_3K_{32} + m_8Q_{47} + B_4Q_{48} + A_4Q_{49} + B_4Q_{50}\right),$$

$$l_{40} = \frac{\beta B_{27}l_{32}}{l_{33}} - \beta A_{27}, \qquad l_{41} = \frac{\beta B_{27}l_{34}}{l_{33}} + l_{38}, \qquad l_{42} = \frac{\beta B_{27}l_{35}}{l_{33}} + l_{39},$$

$$l_{43} = A_5 + l_{40}, \qquad l_{44} = l_{40}l_{36} + l_{41}, \qquad l_{45} = l_{40}l_{37} + l_{42},$$

$$m_1 = \frac{1 + \sqrt{1 + 4m^2}}{2}, \qquad m_2 = \frac{1 - \sqrt{1 + 4m^2}}{2}, \qquad m_3 = \frac{1}{\alpha},$$

$$m_4 = m_1 + m_2, \qquad m_5 = \frac{\Pr}{\beta}, \qquad m_6 = A_1 + m_1,$$

$$m_7 = A_1 + m_2, \qquad m_8 = A_4 + m_3,$$

$$P_1 = A_3C_1 + A_2C_2, \qquad P_2 = \frac{A_2P}{m^2}, \qquad P_3 = \frac{A_3P}{m^2},$$

$$P_4 = -F_1B_1^2XC_9 + F_2B_1XC_{10} + F_3XC_9, \qquad P_5 = -F_1B_1^2XC_{10} - F_2B_1XC_9 + F_3XC_{10},$$

$$P_6 = -F_6B_1^2XC_9 + F_7B_1XC_{10} + F_8XC_9, \qquad P_7 = -F_6B_1^2XC_{10} - F_7B_1XC_9 + F_8XC_{10},$$

$$P_8 = -F_{11}B_1^2XC_9 + F_{12}B_1XC_{10} + F_{13}XC_9, \qquad P_9 = -F_{11}B_1^2XC_{10} - F_{12}B_1XC_9 + F_{13}XC_{10},$$

$$P_{10} = -F_1B_1^2YC_9 + F_2B_1YC_{10} + F_3YC_9, \qquad P_{11} = -F_1B_1^2YC_{10} - F_2B_1YC_9 + F_3YC_{10},$$

$$P_{12} = -F_6B_1^2YC_9 + F_7B_1YC_{10} + F_8YC_9, \qquad P_{13} = -F_6B_1^2YC_{10} - F_7B_1YC_9 + F_8YC_{10},$$

$P_{14} = -F_{11}B_1^2 YC_9 + F_{12}B_1 YC_{10} + F_{13}YC_9$, $P_{15} = -F_{11}B_1^2 YC_{10} - F_{12}B_1 YC_9 + F_{13}YC_{10}$,

$P_{16} = A_1 XC_9 + B_1 XC_{10}$, $P_{17} = A_1 XC_{10} - B_1 XC_9$,

$P_{18} = -F_1 B_1^2 P_{16} + F_2 B_1 P_{17} + F_3 P_{16}$, $P_{19} = -F_1 B_1^2 P_{17} - F_2 B_1 P_{16} + F_3 P_{17}$,

$P_{20} = -F_6 B_1^2 P_{16} + F_7 B_1 P_{17} + F_8 P_{16}$, $P_{21} = -F_6 B_1^2 P_{17} - F_7 B_1 P_{16} + F_8 P_{17}$,

$P_{22} = -F_1 B_1^2 Q_{16} + F_2 B_1 Q_{17} + F_3 Q_{16}$, $P_{23} = -F_1 B_1^2 Q_{17} - F_2 B_1 Q_{16} + F_3 Q_{17}$,

$P_{24} = -F_6 B_1^2 Q_{16} + F_7 B_1 Q_{17} + F_8 Q_{16}$, $P_{25} = -F_6 B_1^2 Q_{17} - F_7 B_1 Q_{16} + F_8 Q_{17}$,

$P_{26} = A_6 + A_{11} - B_{16} + A_{21} - B_{24}$, $P_{27} = A_7 + A_{12} - B_{17} + A_{22} - B_{25}$,

$P_{28} = A_8 + A_{13} - B_{18} + A_{23} - B_{26}$, $P_{29} = A_9 + A_{14} - B_{19}$,

$P_{30} = A_{10} + A_{15} - B_{20}$, $P_{31} = K_{19}(P_4 - Q_{10}) + K_{24}(P_{18} - Q_{22})$,

$P_{32} = K_{19}(P_5 - Q_{11}) + K_{24}(P_{19} - Q_{23})$, $P_{33} = K_{20}(P_6 - Q_{12}) + K_{25}(P_{20} - Q_{24})$,

$P_{34} = K_{20}(P_7 - Q_{13}) + K_{26}(P_{21} - Q_{25})$, $P_{35} = K_{21}(P_8 - Q_{14})$,

$P_{36} = K_{21}(P_9 - Q_{15})$, $P_{37} = A_4 XC_{11} + B_4 XC_{12}$,

$P_{38} = A_4 XC_{12} - B_4 XC_{11}$, $P_{39} = -F_{16}B_4^2 P_{37} + F_{17}B_4 P_{38} + F_{18}P_{37}$,

$P_{40} = -F_{16}B_4^2 P_{38} - F_{17}B_4 P_{37} + F_{18}P_{38}$, $P_{41} = -F_{21}B_4^2 P_{37} + F_{22}B_4 P_{38} + F_{23}P_{37}$,

$P_{42} = -F_{21}B_4^2 P_{38} - F_{22}B_4 P_{37} + F_{23}P_{38}$, $P_{43} = -F_{16}B_4^2 Q_{37} + F_{17}B_4 Q_{38} + F_{18}Q_{37}$,

$P_{44} = -F_{16}B_4^2 Q_{38} - F_{17}B_4 Q_{37} + F_{18}Q_{38}$, $P_{45} = -F_{21}B_4^2 Q_{37} + F_{22}B_4 Q_{38} + F_{23}Q_{37}$,

$P_{46} = -F_{21}B_4^2 Q_{38} - F_{22}B_4 Q_{37} + F_{23}Q_{38}$, $P_{47} = K_{27}(P_{39} - Q_{43})$,

$P_{48} = K_{27}(P_{40} - Q_{44})$, $P_{49} = K_{28}(P_{41} - Q_{45})$,

$P_{50} = K_{28}(P_{42} - Q_{46})$,

$Q_1 = B_3 C_1 + B_2 C_2$, $Q_2 = \dfrac{B_2 P}{m^2}$, $Q_3 = \dfrac{B_3 P}{m^2}$,

$Q_4 = F_4 B_1 XC_{10} + F_5 XC_9$, $Q_5 = -F_4 B_1 XC_9 + F_5 XC_{10}$, $Q_6 = F_9 B_1 XC_{10} + F_{10} XC_9$,

$Q_7 = -F_9 B_1 XC_9 + F_{10} XC_{10}$, $Q_8 = F_{14} B_1 XC_{10} + F_{15} XC_9$, $Q_9 = -F_{14} B_1 XC_9 + F_{15} XC_{10}$,

$Q_{10} = F_4 B_1 YC_{10} + F_5 YC_9$, $Q_{11} = -F_4 B_1 YC_9 + F_5 YC_{10}$, $Q_{12} = F_9 B_1 YC_{10} + F_{10} YC_9$,

$Q_{13} = -F_9 B_1 YC_9 + F_{10} YC_{10}$, $Q_{14} = F_{14} B_1 YC_{10} + F_{15} YC_9$, $Q_{15} = -F_{14} B_1 YC_9 + F_{15} YC_{10}$,

$Q_{16} = A_1 YC_9 + B_1 YC_{10}$, $Q_{17} = A_1 YC_{10} - B_1 YC_9$, $Q_{18} = F_4 B_1 P_{17} + F_5 P_{16}$,

$Q_{19} = -F_4 B_1 P_{16} + F_5 P_{17}$, $Q_{20} = F_9 B_1 P_{17} + F_{10} P_{16}$, $Q_{21} = -F_9 B_1 P_{16} + F_{10} P_{17}$,

$Q_{22} = F_4 B_1 Q_{17} + F_5 Q_{16}$, $Q_{23} = -F_4 B_1 Q_{16} + F_5 Q_{17}$, $Q_{24} = F_9 B_1 Q_{17} + F_{10} Q_{16}$,

$Q_{25} = -F_9 B_1 Q_{16} + F_{10} Q_{17}$, $Q_{26} = B_6 + B_{11} + A_{16} + B_{21} + A_{24}$,

$Q_{27} = B_7 + B_{12} + A_{17} + B_{22} + A_{25}$, $Q_{28} = B_8 + B_{13} + A_{18} + B_{23} + A_{26}$,

$Q_{29} = B_9 + B_{14} + A_{19}$, $Q_{30} = B_{10} + B_{15} + A_{20}$,

$Q_{31} = K_{19}(Q_4 + P_{10}) + K_{24}(Q_{18} + P_{22})$, $Q_{32} = K_{19}(Q_5 + P_{11}) + K_{24}(Q_{19} + P_{23})$,

$Q_{33} = K_{20}(Q_6 + P_{12}) + K_{25}(Q_{20} + P_{24})$, $Q_{34} = K_{20}(Q_7 + P_{13}) + K_{26}(Q_{21} + P_{25})$,

$Q_{35} = K_{21}(Q_8 + P_{14})$, $Q_{36} = K_{21}(Q_9 + P_{15})$, $Q_{37} = A_4 YC_{11} + B_4 YC_{12}$,

$Q_{38} = A_4 YC_{12} - B_4 YC_{11}$, $Q_{39} = F_{19} B_4 P_{38} + F_{20} P_{37}$, $Q_{40} = -F_{19} B_4 P_{37} + F_{20} P_{38}$,

$Q_{41} = F_{24} B_4 P_{38} + F_{25} P_{37}$, $Q_{42} = -F_{24} B_4 P_{37} + F_{25} P_{38}$, $Q_{43} = F_{19} B_4 Q_{38} + F_{20} Q_{37}$,

$Q_{44} = -F_{19} B_4 Q_{37} + F_{20} Q_{38}$, $Q_{45} = F_{24} B_4 Q_{38} + F_{25} Q_{37}$, $Q_{46} = -F_{24} B_4 Q_{37} + F_{25} Q_{38}$,

$Q_{47} = K_{27}(Q_{39} + P_{43})$, $Q_{48} = K_{27}(Q_{40} + P_{44})$, $Q_{49} = K_{28}(Q_{41} + P_{45})$,

$Q_{50} = K_{28}(Q_{42} + P_{46})$,

$r_1 = \sqrt{(1+4m^2)^2 + 16\omega^2}$, $r_2 = \sqrt{1+16\omega^2}$, $r_3 = \sqrt{\Pr^4 + 16\omega^2 \Pr^2}$,

$\theta_1 = \tan^{-1}\left(\frac{4\omega}{1+m^2}\right)$, $\theta_2 = \tan^{-1}(4\omega)$, $\theta_3 = \tan^{-1}\left(\frac{4\omega}{\Pr}\right)$,

$XC_9 = \frac{l_{11} l_{26} - l_{12} B_1}{l_{10} B_1 - l_{25} l_{11}}$, $XC_{10} = \frac{-(l_{10} XC_9 + l_{12})}{l_{11}}$, $XC_{11} = XC_9 + l_{18}$,

$XC_{12} = \frac{-(XC_{11} l_{14} + l_{16})}{l_{15}}$, $XC_{13} = \frac{l_{29} l_{44} - B_5 l_{31}}{B_5 l_{28} - l_{29} l_{43}}$, $XC_{14} = \frac{-(l_{28} XC_{13} + l_{31})}{l_{29}}$,

$$XC_{15} = XC_{13} + l_{36}, \qquad XC_{16} = \frac{-(l_{32} XC_{15} + l_{34})}{l_{33}},$$

$$YC_9 = \frac{l_{11} l_{27} - l_{13} B_1}{l_{10} B_1 - l_{25} l_{11}}, \qquad YC_{10} = \frac{-(l_{10} YC_9 + l_{13})}{l_{11}}, \qquad YC_{11} = YC_9 + l_{19},$$

$$YC_{12} = \frac{-(YC_{11} l_{14} + l_{17})}{l_{15}}, \qquad YC_{13} = \frac{l_{29} l_{44} - B_5 l_{32}}{B_5 l_{28} - l_{29} l_{43}}, \qquad YC_{14} = \frac{-(l_{28} YC_{13} + l_{32})}{l_{29}},$$

$$YC_{15} = YC_{13} + l_{37}, \qquad YC_{16} = \frac{-(l_{32} YC_{15} + l_{35})}{l_{33}}.$$

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