Filtration solutions for variable inputs

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Abstract

Solutions of the deep bed filtration equations are sought for situations involving spatially varying initial porosites and temporally varying superficial velocities and inlet concentrations. Results of a general nature are obtained for both a quasistatic model and a diffusionless model. In the latter case it is shown that one numerical solution can sometimes be used to infer information about a class of problems.

Keywords: Filtration; Modeling; Dynamics

1. Introduction

Most filtration simulations are carried out assuming that the initial porosity of the filter is independent of position and that the superficial velocity and the entrance concentration to which the filter is subjected are independent of time. It is of interest to know the degree to which deviations from these assumptions affect predictions of filter performance. In the present paper several aspects of this question are investigated.

A closed form solution is obtained for a special case of quasistatic filtration with non-negligible diffusivity. This solution holds for arbitrary temporal variations of superficial velocity and entrance concentration, but is restricted to an idealized attachment function.

A general class of problems involving more realistic attachment/detachment functions but negligible diffusivity is investigated. The equations are formulated so that a solution for any given inlet concentration contains information pertinent to all associated superficial velocities and initial porosities. Two special cases of this general class are then singled out for further investigation.

The popular class of attachment functions discussed by Tien [1] is shown to lead to a formulation allowing a single solution to be representative of all possible variations of inlet concentrations, superficial velocities and initial porosities. Some specific examples are given.

Linear 'chromatography like' attachment/detachment functions (see, for instance, Tien [1] or Adin and Rajagopalan [2]) are also considered. Numerical solutions are obtained by the method of characteristics and it is shown how one such solution can be used in several different ways.

It should be pointed out that filtration models differ mainly in the forms of the attachment/detachment functions employed therein. The complexity of these functions is determined by the number of phenomena they attempt to describe. The simplest models are those involving pure attachment such as that proposed by Iwasaki [3]. These models can be modified to account for filter clogging in a number of ways. The papers listed in Table 2.1 of Tien [1] are representative of this work. Soo and Radke [4] and Soo et al. [5] showed that it was possible to use such models to describe attachment by simultaneous straining and interception. Models allowing for both attachment and detachment (re-entrainment of previous captured particles) have been developed. Repre-

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sentative of this work are the papers by Adin [6], Adin and Rebhum [7], and Adin and Rajagopalan [2]. As discussed in the latter paper and by Rajagopalan and Chu [8], it is possible to reinterpret such models to describe the reduction in attachment rate associated with the covering of filter grain surfaces. Privman et al. [9] dealt with this phenomenon explicitly. Song and Elimelech [10] extended this work by introducing an internal variable which describes the time evolution of the surface covering. It is interesting to note that all these models except the one associated with pure attachment predict s-shaped breakthrough curves which are qualitatively similar.

It appears that use of internal variables is a promising approach to description of a number of filtration phenomena and should be pursued further in the future. The present work, however, deals only with filtration theories in which internal variables are absent. It is believed that the general attachment/detachment models employed herein include most of the approaches to attachment, clogging, and detachment discussed above.

2. Governing equations

All work in the present paper will be based on the one-dimensional deep filtration equations reviewed by Tien [1]. These can be written in the dimensionless forms

$$N_2 \epsilon_i \partial_t c + q \partial_z c = N_3 \partial_z (D \partial_z c) - N_1 M, \partial_t s = N_1 M$$

$$c(0,t) = c_i, c(z,0) = 0, s(z,0) = 0 \tag{1}$$

$$E = 1 - c_0/c_i; \quad c_0 = c(1,t)$$

where

$$N_1 = \lambda L, \quad N_2 = \hat{\epsilon} L/(\hat{q} T_p), \quad N_3 = \hat{D}/(\hat{q} L) \tag{2}$$

are dimensionless numbers which characterize the process. In particular, N_1 measures the influence of attachment/detachment, N_2 the influence of disturbance front propogation, and N_3 the influence of diffusion. The functions $\epsilon_i(z)$, $q(t)$, c_i (t), $D(c,s,z,t)$, and $M(c,s,z,t)$ must be specified in each case to create a determinate system of equations. In [1] and [2] L is the filter length; T_p is the process time; λ is the characteristic inverse filtration length; and $\hat{c}, \hat{D}, \hat{q}$ and $\hat{\epsilon}$ are respective characteristic values of concentration, diffusivity, superficial velocity and porosity. In addition, Lz denotes distance from the filter entrance, $T_p t$ denotes time, $\hat{c}c$ denotes concentration, $\hat{c}c_i$ denotes entrance concentration, $\hat{q}\hat{c}T_p s/L$ denotes specific deposit, $\hat{\epsilon}\epsilon_i$ denotes initial porosity, $\hat{D}D$ denotes diffusivity, $q\hat{c}\lambda M$ denotes attachment rate, and qq denotes superficial velocity.

Equations (1) are general enough to cover a variety of situations but not all. As mentioned earlier, theories containing internal variables (such as the one proposed by Song and Elimelech [10]) would require the addition of the evolution equations for these internal variables to (1). For many industrial filtration conditions (see, for instance, Tien [1]) the inequalities $N_2 \ll 1$ and $N_3 \ll 1$ are satisfied and the terms multiplied by these quantities can be safely neglected. For other cases such as the transport and collection of bacteria in soils (see, for instance Corapcioglu et al. [11] or Abboud and Corapcioglu [12]) the above mentioned terms are important and must be maintained.

It is, of course, possible to solve (1) numerically for many forms of $\epsilon_i(z)$, $c_i(t)$, $q(t)$, $D(c,s,z,t)$, and $M(c,s,z,t)$. Here, to facilitate analytical work, attention will be restricted to some interesting special cases.

3. Quasistatic model

The inequality $N_2 \ll 1$ is satisfied in many practical situations. In the present work the terminology quasistatic model will denote the special case of (1) obtained by formally equating N_2 to zero. It is clear that all quasistatic predictions are independent of the initial porosity distribution. It does not appear possible to obtain many general results from the quasistatic model. Here attention is focused on the most idealized attachment and diffusion models (see, for instance, Tien [1]) in order to produce one general solution.

Substituting

$$N_2 = 0, \quad D = 1, \quad M = qc \tag{3}$$

into (1) yields

$$\partial_z c = -N_1 c + N_3 \partial_{zz} c/q(t), \quad \partial_t s = N_1 q(t)c$$

$$c(0,t) = c_i(t), s(z,0) = 0 \tag{4}$$

$$E = 1 - c(1,t)/c_i(t)$$

Equations (4) are readily solved to yield

$$c = c_i(t) \exp\left(-\left((1 + 4N_1 N_3/q(t))^{1/2} - 1\right) q(t)z/(2N_3)\right)$$

$$s = N_1 \int_0^t c_i(\hat{t}) q(\hat{t}) \exp\left(-\left((1 + 4N_1 N_3/q(\hat{t}))^{1/2} - 1\right)\right)$$

$$\times q(\hat{t})z/(2 N_3)\right) d\hat{t} \tag{5}$$

$$E = 1 - \exp\left(-\left((1 + 4N_1 N_3/q(t))^{1/2} - 1\right) q(t)/(2N_3)\right)$$

These results hold for arbitrary temporal variations of inlet concentration and superficial velocity. In particu-

lar, the efficiency is independent of the inlet concentration. The price which must be paid for this generality is the restrictiveness embodied in (3b,c).

The symbol M represents the combined effects of attachment and detachment. The attachment rate will normally vary with specific deposit. Equation (3a) does not allow for this or for the detachment phenomenon. It is the idealized pure attachment model proposed by Iwasaki [3] and extended to account for grain covering by Privman et al. [9].

The symbol D represents the sum of molecular and turbulent diffusivity. For situations in which this quantity is not negligible, turbulence is likely to make an important contribution and cause D to be variable. Equation (3c) cannot represent this effect.

4. Diffusionless model

The inequality $N_3 < < 1$ is assumed to be satisfied in most filtration simulations. The terminology diffusionless model will denote herein the special case of (1) obtained by formally equating N_3 to zero. It appears that several results of a general nature follow from this model. Substituting

$$N_3 = 0, \quad M = q\, P(c,s) \tag{6}$$

into (2 a–d) and writing the resulting equations in terms of

$$\theta = \theta_t(t) - N_2\theta_z(z); \quad \theta_t = \int_0^t q(\hat{t})\, d\hat{t}, \quad \theta_z = \int_0^z \epsilon_i(\hat{z})\, d\hat{z} \tag{7}$$

(rather than t) leads to

$$\begin{gathered} \partial_z c = -N_1 P(c,s), \quad \partial_\theta s = N_1 P(c,s) \\ c(0,\theta) = c_i(\theta), s(z,0) = 0 \end{gathered} \tag{8}$$

Equating θ to zero in (7) yields the equation for the boundary between the undisturbed and disturbed regions (leading characteristic). This equation can be written in the symbolic form

$$z = z_0(t) \tag{9}$$

It should be pointed out that θ_t is a generalization of the modified time variables employed by such investigators as Adin and Rajagopalan [2] and Soo and Radke [4] while θ is a generalization of the reduced time variables used by such authors as Tien [1] and Soo and Radke [4].

Equation (7) evaluated at $z = 0$ can be used to evaluate $t(\theta)$ at $z = 0$ for a given $q(t)$. This can be used together with a given $c_i(t)$ to determine $c_i(\theta)$.

Once this is done, (8) can be solved to find $c(z,\theta)$ and $s(z,\theta)$. These are used in connection with (7) to determine $c(z,t)$ and $s(z,t)$. The results should be used only in the disturbed region $z < z_0(t)$.

It is to be noted that one solution for $c(z,\theta)$ and $s(z,\theta)$ determined by the methodology discussed above contains information pertinent to all initial porosity distributions. Furthermore, in the special case of constant inlet concentration one such solution contains information pertinent to all initial porosity and superficial velocity variations.

It is of interest to examine two special cases of the diffusionless model formulated in this section. Each of the next two sections is devoted to one of these special cases.

5. General attachment model

One of the most popular attachment/detachment models in applications (see, for instance, Tien [1] and the references in Table 2.1 therein, Soo and Radke [4] and Soo et al. [5]) is the attachment/clogging model

$$P = cQ(s) \tag{10}$$

Substituting (10) into (8) and writing the resulting equations in terms of the independent variables z and

$$\tau = \int_0^\theta c_i(\hat{\theta})\, d\hat{\theta} \tag{11}$$

and the dependent variables s and

$$r = c/c_i(\theta) \tag{12}$$

yields

$$\begin{gathered} \partial_z r = -N_1 r\, Q(s), \quad \partial_\tau s = N_1 r\, Q(s) \\ r(0,\tau) = 1, \, s(z,0) = 0 \end{gathered} \tag{13}$$

These equations are well known and have been written in terms of τ and r to emphasize the fact that (13) is formally identical to the corresponding quasistatic problem involving constant inlet concentration, superficial velocity and initial porosity. One solution of (13) contains information about all variations of these quantities.

The general solution to (13) can be written in the implicit form (see, for instance, Tien [1])

$$\int_0^{s_i} d\hat{s}_i/Q(\hat{s}_i) = N_1\tau, \quad \int_{s_i}^s d\hat{s}/(\hat{s}Q(\hat{s})) = -N_1 z, \, r = s/s_i \tag{14}$$

Here (14a) is used to find $s_i(\tau) = s\,(0,\tau)$, then (14b) is used to find $s(z,\tau)$, and finally (14c) is used to find

$r(z,\tau)$. Once these calculations are completed (7), (11) and (12) can be used to find the corresponding forms of $s(z,t)$ and $c(z,t)$. In the special case of constant inlet concentration ($c_i = 1$) it is interesting to note that (7) and (14) can be combined to yield

$$\tau = \theta_t(t) - N_2\theta_z(z) \tag{15}$$

while (12) simplifies to

$$r = c \tag{16}$$

For most forms of $Q(s)$ the integrals in (14) must be carried out numerically (or, alternatively, direct numerical solutions of (13) can be performed). In a few cases closed form solutions are possible. As an example consider

$$Q = 1 \Big/ \Big(1 + \sum_{j=1}^{J} k_j\, s^{\lambda_j}\Big) \tag{17}$$

where J is a positive integer, the k's are constants, and the λ's are positive constants. Then (14 a,b) can be integrated to yield

$$s_i + \sum_{j=1}^{J} k_j\, s_i^{1+\lambda_j}/(1+\lambda_j) = N_1\tau$$
$$\ln(s_i/s) + \sum_{j=1}^{J} k_j(s_i^{\lambda_j} - s^{\lambda_j})/\lambda_j = N_1 z \tag{18}$$

This one solution can be to be employed to generate numerous results involving variable inlet concentrations, superficial velocities, and initial porosities.

6. Linear attachment/detachment model

A generalization of (10) to allow for both attachment and detachment phenomena is (see, for instance, Tien [1], Adin and Rajagopalan [2], Adin [6], Adin and Rebhun [7], and Rajagopalan and Chu [8])

$$P = -Q_0(s) + cQ_1(s) \tag{19}$$

In the present work attention will be focused on the special case of $Q_0 = N_4 s/N_1$ (N_4 being constant) and $Q_1 = 1$. This produces the linear 'chromatography like' model

$$P = -N_4 s/N_1 + c \tag{20}$$

Substituting (20) into (8) yields

$$\partial_z c = N_4 s - N_1 c, \partial_\theta s = N_1 c - N_4 s$$
$$c(0,\theta) = c_i(\theta), s(z,0) = 0 \tag{21}$$

In the context of (21) N_1 plays the role of an attach-
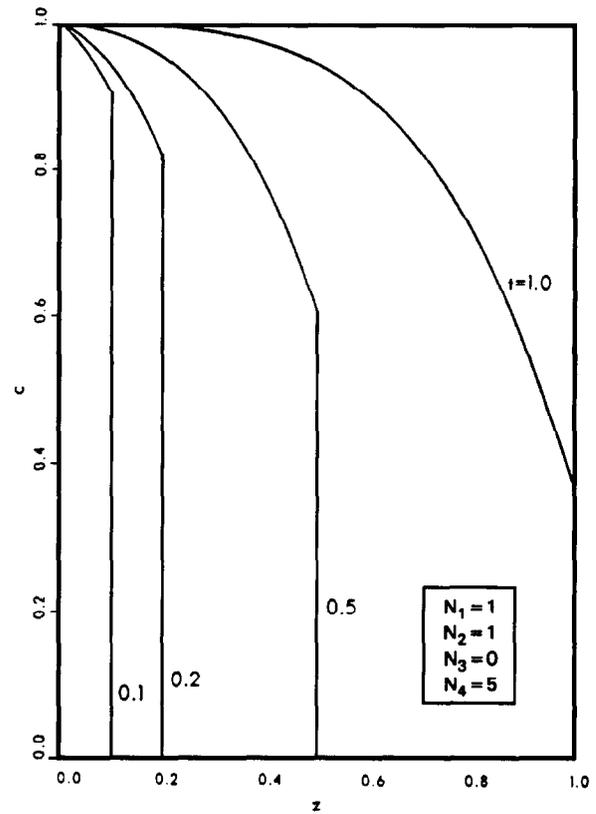

Fig. 1. Temporal evolution of concentration profiles predicted by linear attachment/detachment model.
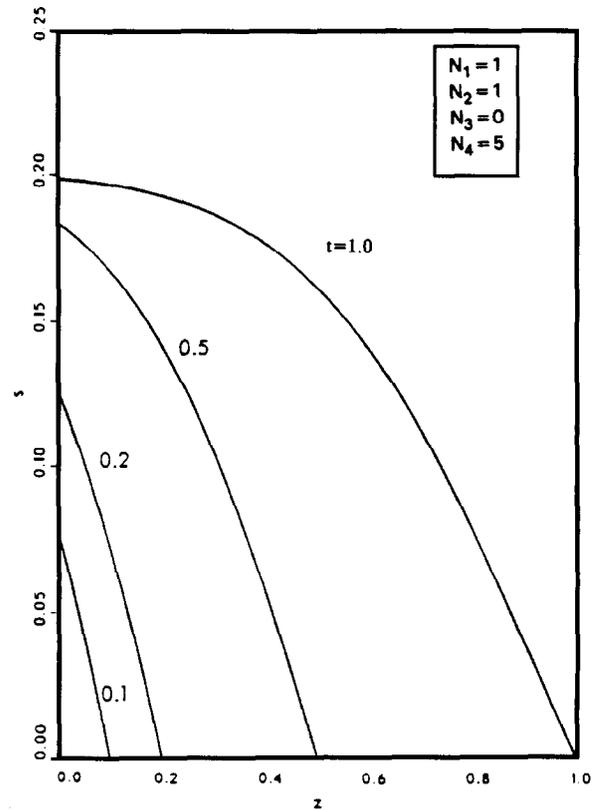

Fig. 2. Temporal evolution of specific deposit profiles predicted by linear attachment/detachment model.

ment constant while N_4 plays the role of a detachment constant.

The problem consisting of (21) has a closed form solution (see, for instance, Adin and Rajagopalan [2]) but, because of its complicated nature, it will not be repeated here. Symbolically it can be expressed as

$$c(z,\theta) = C(z,\theta), s(z,\theta) = S(z,\theta) \tag{22}$$

Combining (7) and (21) leads to

$$c(z,t) = C(z,\theta_t(t) - N_2 \theta_z(z)),$$
$$s,(z,t) = S(z,\theta_t(t) - N_2 \theta_z(z)) \tag{23}$$

where (23) is used only in the disturbed region $z < z_0$ (t).

While the exact expressions for $C(z,\theta)$ and $S(z,\theta)$ are quite complicated, the method of Rasmussen [13] can be used to determine an appproximate solution valid for moderate and large values of θ which is algebraically tractable. For the important special case of constant inlet concentration ($c_i = 1$) the approximate expression for the function C reads

$$C(z,\theta) = (\text{erfc}((N_4/\theta)^{1/2}(N_1 z/N_4 - \theta)/2)$$
$$+ \exp(N_1 z)\text{erfc}((N_4/\theta)^{1/2}$$

$$(N_1 z/N_4 + \theta)/2))H(\theta)/2 \tag{24}$$

where H denotes the unit step function.. Making the indicated transformations produces

$$c(z,t) = (\text{erfc}((N_4/(\theta_t(t) - N_2 \theta_z(z)))^{1/2}$$
$$\times(N_1 z/N_4 + N_2 \theta_z(z) - \theta_t(t))/2)$$
$$+ \exp(N_1 z)\text{erfc}(N_4/(\theta_t (t) - N_2 \theta_z(z))^{1/2}$$
$$\times(N_1 z/N_4 - N_2 \theta_z(z)$$
$$+ \theta_t(t))/2))H(\theta_t(t) - N_2\theta_z(z))/2 \tag{25}$$

This expression holds for all initial porosites and superficial velocities.

The expressions (23) can be used both to obtain results of a general nature and to compute specific solutions. One example of each use will be given below.

Consider the expressions

$$q = 1 + \alpha \cos(2\pi t/T), \ \epsilon_i = 1 + \beta \cos(2\pi z/Z) \tag{26}$$

where α and β are constants and T and Z are positive constants. Equations (26) provide an opportunity to investigate the effect of perturbations in the
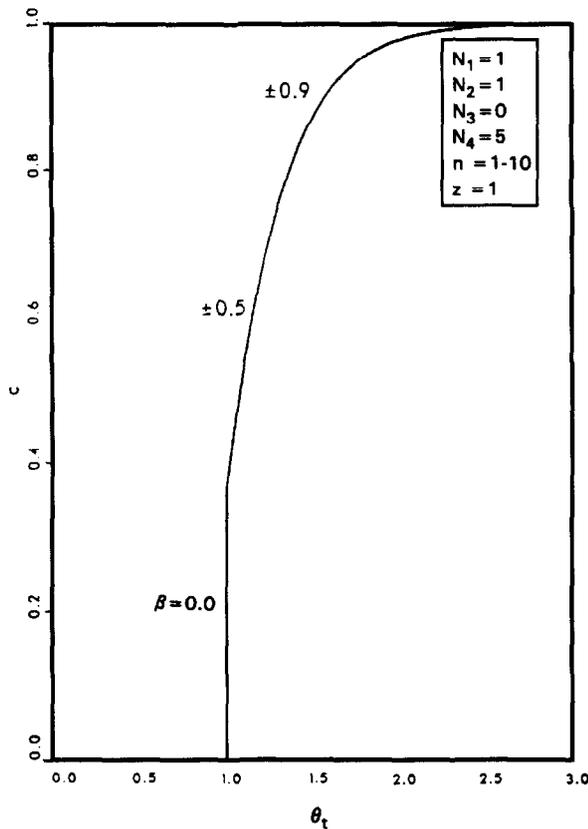

Fig. 3. Outlet concentration histories predicted by linear attachment/detachment model for five initial porosity variations.
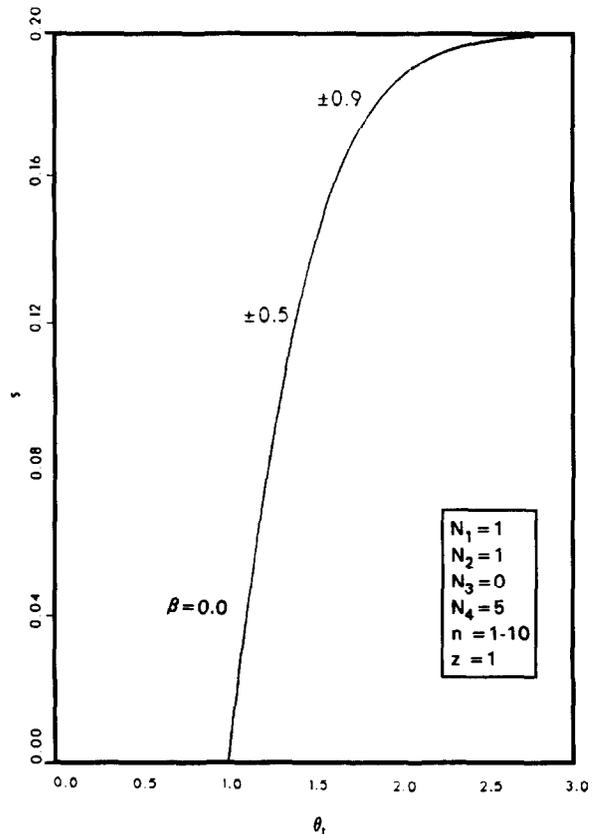

Fig. 4. Outlet specific deposit histories predicted by linear attachment/detachment model for five initial porosity variations.

superficial velocity and initial porosity which vary periodically about mean values. Substituting (26) into (7) yields

$$\theta = t - N_2 z + (\alpha T \sin(2\pi t/T)$$
$$- N_2 \beta Z \sin(2\pi z/Z))/(2\pi) \qquad (27)$$

Equation (26a) indicates that the superficial velocity has a unit average over a time interval $T/2$. Equation (26b) indicates that the initial porosity has a unit average over a distance interval $Z/2$. As these intervals decrease (27) approaches the expression $\theta = t - N_2 z$ which holds for constant superficial velocity and constant initial porosity. The influence of the fluctuations about mean values inherent in (26a,b) is, therefore, reduced by increasing the rapidity of the fluctuations. This is a general result which is independent of the specific form of $c_i(t)$.

Expressions (23) can be used in a variety of ways to obtain numerical solutions for specific cases. For constant superficial velocity ($q = 1$) and constant initial porosity ($\epsilon_i = 1$) (23) reduces to

$$c(z,t) = C(z,t - N_2 z)H(t - N_2 z),$$
$$s(z,t) = S(z,t - N_2 z)H(t - N_2 z) \qquad (28)$$

Figs. 1 and 2 show some typical concentration and specific deposit profiles found by solving (21) numerically by the method of characteristics (this was felt to be easier than numerically evaluating the exact solution) for the special case of $c_i = 1$ to determine $C(z,\theta)$ and $S(z,\theta)$ and using (28). Results can be obtained in a similar way for any choice of $c_i(t)$.

If $\epsilon_i(z)$ (and, thus, $\theta_z(z)$) is specified, plots of c and s vs. $\theta_i(t)$ for a fixed z will hold for any $q(t)$. Figs. 3–6 show some typical results of this kind for $c_i = 1$ using (26b) with

$$Z = 2/n \qquad (29)$$

(n being a positive integer). Employing (29) insures that the average porosity in the filter is unity. It was shown by Allaham et al. [14] that the outlet conditions (and thus the efficiency) are the same for all porosity distributions having the same average.

Figs. 3 and 4 confirm this for several values of n and β. Fig. 3 is a typical breakthrough curve. When the normalized exit concentration 'breaks through' to unity the collection efficiency of the filter becomes zero. In the interior of the filter the solution depends on both n and β. Figs. 5 and 6 show the dependence on β for $n = 1$. As indicated in the discussion of
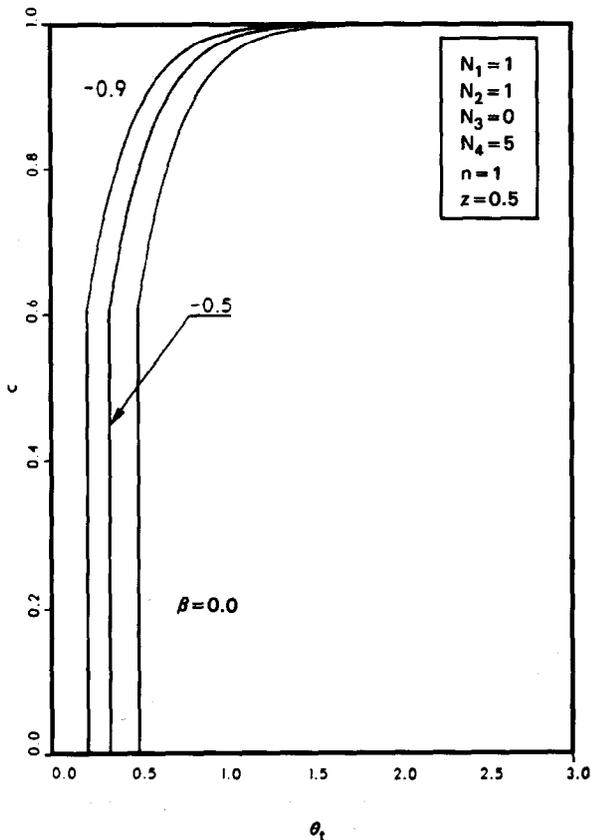

Fig. 5. Center concentration histories predicted by linear attachment/detachment model for three initial porosity variations.
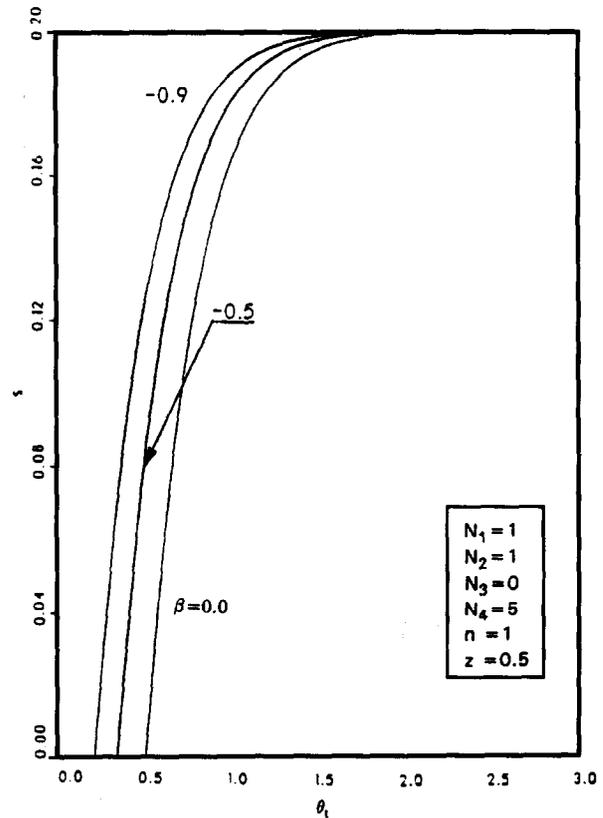

Fig. 6. Center specific deposit histories predicted by linear attachment/detachment model for three initial porosity variations.
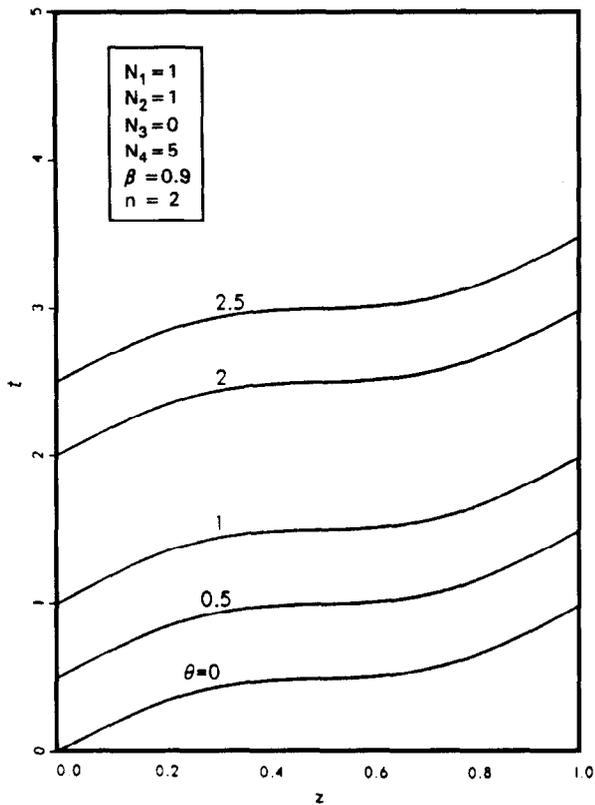
Fig. 7. Characteristic shapes in z,t plane predicted by linear attachment/detachment model.
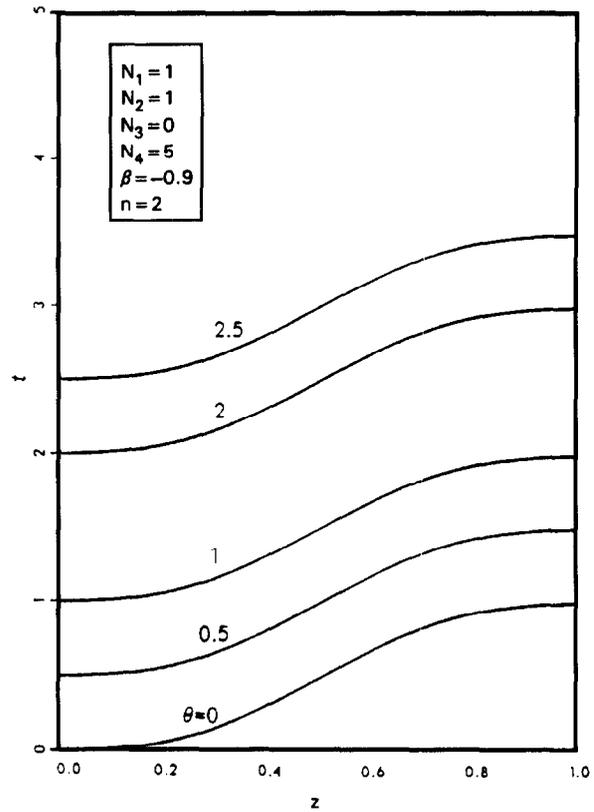

Fig. 8. Characteristic shapes in z,t plane predicted by linear attachment/detachment model.

(27), Figs. 5 and 6 are indicative of the maximum influence of β. As n increases the influence of β will decrease.

Figs. 3-6 hold for any form of $q(t)$. For the special case of $q = 1$ the combination of (27) and (29) reduces to

$$\theta = t - N_2(z + \beta \sin(n\pi z)/(n\pi))$$ (30)

Some typical plots of the characteristics (lines of $\theta =$ constant) in the z,t plane are shown in Figs. 7-10. The magnitude of deviation from the straight line $\theta = t - N_2 z$ decreases as n increases as expected. It is to be noted that in the methodology under discussion herein the characteristics shown in Figs. 7-10 are not used to carry out the numerical solution. This is done in the z, θ plane where both sets of characteristics are straight lines.

Discussion in the present section has been based on the attachment/detachment function (20) for the sake of concreteness. It is clear, however, that similar results hold for any attachment/detachment function of the form (19) or the still more general form (6).

7. Conclusion

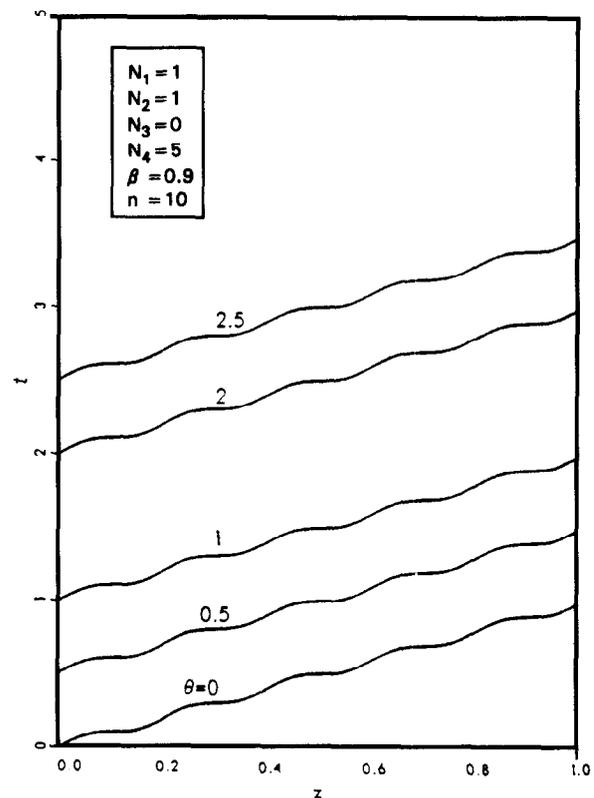In the foregoing the problem of determining solu-

Fig. 9. Characteristic shapes in z,t plane predicted by linear attachment model.
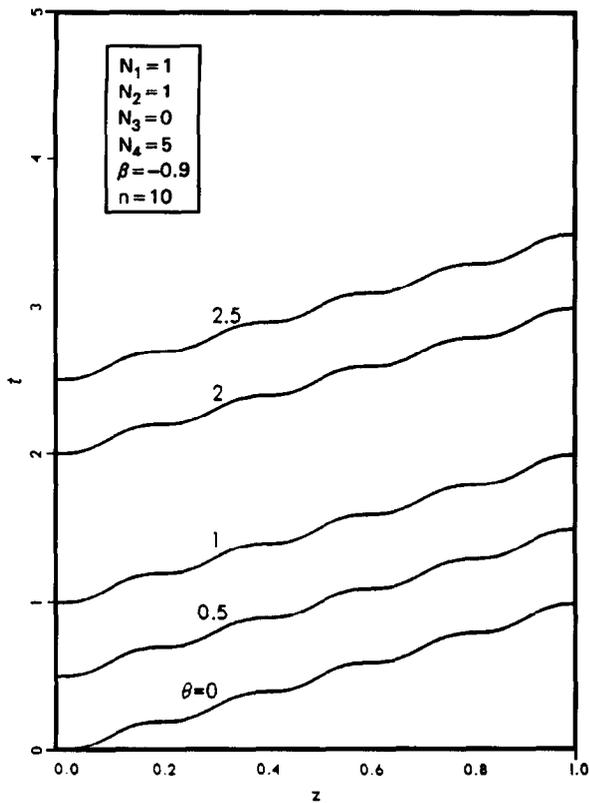
Fig. 10. Characteristic shapes in z,t plane predicted by linear attachment/detachment model.

tions to the filtration equations for situations involving variable inlet concentrations, superficial velocities, and initial porosities was considered. Several situations in which closed form results of general validity could be found were identified and discussed. Specific results, both analytical and numerical, were obtained in a number of cases and a selection of the numerical predictions was presented graphically. It was demonstrated that the information reported could be used to minimize the necessity for numerical computation.

Notation

C	solution of (21)
c	normalized concentration
$\hat{c}$	characteristic concentration
c_i	normalized inlet concentration
c_0	normalized outlet concentration
D	normalized diffusivity
$\hat{D}$	characteristic diffusivity
E	filter efficiency
H	unit step function
L	filter length
M	normalized attachment/detachment function
N_1	dimensionless attachment number
N_2	dimensionless transient completion time number
N_3	dimensionless diffusion number
N_4	dimensionless detachment number
n	integer
P	attachment function
Q	attachment function
Q_0	detachment function
Q_1	attachment function
q	normalized superficial velocity
$\hat{q}$	characteristic superficial velocity
r	concentration ratio
S	solution of (21)
s	normalized specific deposit
s_i	normalized entrance specific deposit
T	normalized period of superficial velocity variation
T_p	process time
t	normalized time
$\hat{t}$	dummy variable
Z	length of initial porosity variation
z	normalized position
$\hat{z}$	dummy variable
z_0	normalized length of disturbed region
α	normalized magnitude of superficial velocity variation
β	normalized magnitude of initial porosity variation
$\hat{\epsilon}$	characteristic porosity
ϵ_i	normalized initial porosity
k_j	material constants
λ	characteristic inverse filtration length
λ_j	material constants
θ	reduced modified time
θ_t	modified time
θ_z	modified position
$\hat{\theta}$	dummy variable
τ	modified time

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