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Transient free convection flow of a viscoelastic fluid over vertical surface*

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Abstract In this paper, the viscoelsatic boundary layer flow and the heat transfer near a vertical isothermal impermeable surface and in a quiescent fluid are examined. The governing equations are formulated and solved numerically using MackCormak's technique. The results show excellent agreement with previously published results by a comparision. Representative results for the velocity and temperature profiles, boundary layer thicknesses, Nusselt numbers, and local skin friction coefficients are shown graphically for different values of viscoelsatic parameters. In general, it is found that the velocities increase inside the hydrodynamic boundary layers and the temperatures decrease inside the thermal boundary layers for the viscoelsatic fluid as compared with the Newtonian fluid due to favorable tensile stresses. Consequently, the coefficients of friction and heat transfer enhance for higher viscoelsatic parameters.

Key words viscoelastic flows, transient, free convection heat transfer

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Nomenclature

$\boldsymbol{A}_1$, $\boldsymbol{A}_2$,	first two Rivlin-Ericksen tensors;
C_{f},	local coefficient of friction;
C_{p},	specific heat of the fluid at a constant pressure;
g,	acceleration magnitude for gravity;
Gr,	Grashof number, $\frac{g\beta(T_{\mathrm{w}}-T_\infty)L^3}{\upsilon^2}$;
h,	heat transfer coefficient;
k,	thermal conductivity;
k_0,	elastic parameter;
k_l^*,	dimensionless viscoelsatic parameter, $\frac{k_0 Gr^{\frac{1}{2}}}{L^2}$;
L,	characteristic length of plate;
Nu_x,	local Nusselt number;
$\boldsymbol{PI}$,	spherical stress;

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Pr, Prandtl number, $\frac{\upsilon}{\alpha}$;
τ, dimensionless time;
T, temperature;
T_{w}, wall temperature;
T_∞, ambient fluid temperature;
u, v, dimensionless velocities along the x- and y-axes, respectively;
x, y, dimensionless coordinates.

Greek symbols

α, thermal diffusivity;
α_1, α_2, material moduli;
β, coefficient of thermal expansion;
Θ, non-dimensional temperature;
μ, dynamic viscosity;
υ, kinematic viscosity;
ρ, fluid density;
$\mathbf{\Gamma}$, Cauchy stress tensor.

Subscripts

w, wall surface;
∞, free stream condition.

Superscripts

\- dimensional variables.

1 Introduction

Numerous applications of viscoelsatic fluids in several manufacturing processes have led to renewed interest among researchers to investigate the viscoelsatic boundary layer flow over a stretching plastic sheet[1–8]. Some of the physical applications of such investigations are polymer sheet extrusion from a dye, glass fiber, paper production, and drawing of plastic films. The viscoelsatic fluid model used by Bird et al.[9] was a simplified version of the so-called second-grade fluid. Sakiadis[10] related on the boundary layer approximation proposed by Schlichting[11] for simplifying the governing equations. The final equation is in the form of a forth-order non-linear ordinary differential equation that can not be solved analytically or even numerically due to the lack of sufficient boundary conditions. Various authors[3–8] suggested different simple reduced equations to describe the viscoelastic fluid behaviors by using the perturbation technique proposed by Van Dyke[12].

More recently, Shawaqfah et al.[13] studied numerically the forced convection of the Blasius flow of a “second-grade” visco-elastic fluid, in which a model was selected and the forced convection heat transfer coefficient values were predicted. It was found that the velocities inside the hydrodynamic boundary layers decreased and temperatures increased due to favorable tensile stresses. This decreased both the coefficients of friction and heat transfer rates. Damseh et al.[14] studied the transient mixed convection flow of a second-grade viscoelastic fluid over vertical surfaces. It was found that the velocity decreased inside the boundary layer as the viscoelsatic parameter increased, and the local Nusselt number decreased consequently. This was due to the higher tensile stresses between the viscoelsatic fluid layers which had a retardation effects on the motion of these layers, and consequently, on the heat transfer rates for the mixed convection heat transfer problem under the investigation.

Since in reality, most of the fluids considered in industrial applications are non-Newtonian in nature, especially of viscoelsatic types, we extend the natural convection heat transfer work to viscoelsatic fluid flow and heat transfer. The governing equations for this investigation are written in dimensionless forms using a set of dimensionless variables and solved numerically using MackCormak’s technique. Numerical results for the velocity and temperature profiles as well as the local coefficient of friction and local Nusselt number under the effect of viscoelsatic parameter are presented.

2 Mathematical formulation

The constitutive equation, which satisfies the second-order fluid, is given by Coleman and Noll[15] as

$$\boldsymbol{\Gamma} = -\boldsymbol{PI} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_3, \tag{1}$$

where $\boldsymbol{\Gamma}$ is the Cauchy stress tensor, $-\boldsymbol{PI}$ is the spherical stress due to the constraint of incompressibilities, μ is the dynamics viscosity, α_1 and α_2 are the material moduli, and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{A}_3$ are the kinematics tensors. The model of Eq. (1) was derived by considering up to the second-order approximation of retardation parameters[15]. Fosdick and Rajagopal[16] have shown, by using the data reducting from experiments, that in the case of a second-order fluid, the following relation should holds:

$$\mu \geqslant 0, \quad \alpha_1 \leqslant 0, \quad \alpha_1 + \alpha_2 \neq 0. \tag{2}$$

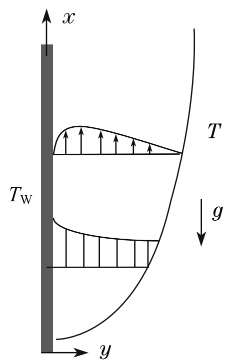

Fig. 1 Transient free convection model for a viscoelastic fluid near a vertical wall

Consider a laminar free convection boundary layer flow of a viscoelsatic fluid over an isothermal vertical flat plate, which is heated in an unsteady manner. The problem is described in a rectangular coordinate system attached to the plate such that the x-axis lies along the plate surface and the y-axis is normal to the plate as shown in Fig. 1. It is assumed that at time $\bar{t} \leqslant 0$, the temperatures of the plate and the viscoelsatic fluid are maintained at the constant temperature T_∞ and at time $\bar{t} > 0$, and the temperature of the plate is impulsively increased to the constant value $T_{\rm w}$ satisfying $T_{\rm w} > T_\infty$. The continuity, momentum, and energy equations under the boundary layer and Boussinesq approximations can be written as[17]

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{3}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g\beta(T - T_\infty) - k_0\left(\bar{u}\frac{\partial^3 \bar{u}}{\partial \bar{x}\partial \bar{y}^2} + \bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} - \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right), \tag{4}$$

$$\frac{\partial T}{\partial \bar{t}} + \bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \alpha\frac{\partial^2 T}{\partial \bar{y}^2}. \tag{5}$$

Here, $\bar{u}$ and $\bar{v}$ are the velocity components in the x- and y- directions, respectively, υ is the kinematic coefficient of viscosity, $\alpha = \frac{k}{\rho c_{\rm p}}$ is the thermal diffusivity, g is the gravitational acceleration, β is the coefficient of thermal expansion, and $k_0 = -\frac{\alpha_1}{\rho}$ is the elastic parameter.

Hence, in the case of a second-order fluid, k_0 takes positive values as α_1. In deriving Eq. (4), it is assumed that the normal stress is of the same order of magnitude of the shear stress in addition to the boundary layer approximations. Equation (5) is the thermal boundary layer equation, and no correction of viscoelsatic behaviors is added on. This is due to the neglect of the viscous dissipation effect.

A critical review on the boundary conditions and the existence and uniqueness of the solution has been given by Rajagopal et al.[1]. Most of available literatures on the boundary layer flow of a viscoelsatic over linearly stretching sheets deal with the three boundary conditions on velocities, which are one less than the number required to solve the problem uniquely. Rollins and Vajravelu[18] derived a unique solution to the problem containing exponential terms of similarity variables. In view of the above discussions on boundary conditions, the physical initial and boundary conditions for this problem are given by

$$\begin{cases} \bar{t} \leqslant 0, & \bar{u}=0,\ \bar{v}=0,\ T=T_\infty \quad \text{for} \quad \bar{x}\geqslant 0,\ \bar{y}\geqslant 0, \\ \bar{t} > 0, & \begin{cases} \bar{u}=0,\ \bar{v}=0,\ T=T_\infty & \text{for} \quad \bar{x}=0,\ \bar{y}\geqslant 0, \\ \bar{u}=0,\ \bar{v}=0,\ T=T_{\rm w} & \text{for} \quad \bar{y}=0,\ \bar{x}\geqslant 0, \\ \bar{u}=0,\ \dfrac{\partial \bar{u}}{\partial \bar{y}}=0,\ T=T_\infty & \text{for} \quad \bar{y}\to\infty, \end{cases} \end{cases} \tag{6}$$

where $(\frac{\partial \bar{u}}{\partial \bar{y}})_{\bar{y}\to\infty}=0$ is taken as a boundary layer condition in order to determine the boundary layer thicknesses. Define the non-dimensional variables such that

$$t=\frac{Gr^{-\frac{1}{2}}v\bar{t}}{L^2},\quad x=\frac{\bar{x}}{L},\quad y=\frac{Gr^{\frac{1}{4}}\bar{y}}{L}, \tag{7}$$

$$u=\frac{Gr^{-\frac{1}{2}}L\bar{u}}{v},\quad v=\frac{Gr^{-\frac{1}{4}}L\bar{v}}{v},\quad \Theta=\frac{T-T_\infty}{T_{\rm w}-T_\infty}, \tag{8}$$

where L is the characteristic length of the plate, and $Gr=\frac{g\beta(T_{\rm w}-T_\infty)L^3}{v^2}$ is the Grashof number. Then, substituting Eqs. (7) and (8) into Eqs. (3)–(6) yields the following dimensionless equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}-k_l^*\left(u\frac{\partial^3 u}{\partial x\partial^2 y}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)+\Theta, \tag{10}$$

$$\frac{\partial \Theta}{\partial t}+u\frac{\partial \Theta}{\partial x}+v\frac{\partial \Theta}{\partial y}=\frac{1}{Pr}\frac{\partial^2 \Theta}{\partial y^2}, \tag{11}$$

where $k_l^*=\frac{k_0 Gr^{\frac{1}{2}}}{L^2}$ is the modified viscoelsatic parameter, and $Pr=\frac{\mu c_{\rm p}}{k}$ is the Prandtl number. Note that for the special case of $k_l^*=0$, the fluid is again a Newtonian fluid. The corresponding dimensionless initial and boundary conditions can be written as

$$\begin{cases} t \leqslant 0, & u=0,\ v=0,\ \Theta=0 \quad \text{for} \quad x\geqslant 0,\ y\geqslant 0, \\ t > 0, & \begin{cases} u=0,\ v=0,\ \Theta=0 & \text{for} \quad x=0,\ y\geqslant 0, \\ u=0,\ v=0,\ \Theta=1 & \text{for} \quad y=0,\ x\geqslant 0, \\ u=0,\ \dfrac{\partial u}{\partial y}=0,\ \Theta=0 & \text{for} \quad y\to\infty. \end{cases} \end{cases} \tag{12}$$

The dimensionless skin friction coefficient $C_{\rm f}$ and local Nusselt number Nu_x are important physical parameters for this type of flow and heat transfer situation[19–20]. They can be defined in dimensionless forms as

$$C_{\rm f}Gr^{\frac{3}{4}} = \Big(\frac{\partial u}{\partial y}\Big)_{(x,\,0,\,t)} - 2k_l^*\Big(\frac{\partial u}{\partial y}\Big)_{(x,\,0,\,t)}\Big(\frac{\partial v}{\partial y}\Big)_{(x,\,0,\,t)}, \tag{13}$$

$$Nu_xGr^{-\frac{1}{4}} = -\Big(\frac{\partial \Theta}{\partial y}\Big)_{(x,\,0,\,t)}. \tag{14}$$

3 Results and discussions

The transient boundary layer equations represented by Eqs. (9)–(11) are solved subject to the initial and boundary conditions given by Eq. (12) using McCormack's method, which is an explicit finite-difference technique of the second-order accuracy in space and time. The details of this method of solution were clearly explained by Anderson[21]. The employed numerical solution was a time marching technique giving the downstream velocity and temperature profiles with known upstream profiles. In the present work, the above quantities are calculated by obtaining explicitly the flow field variables at grid point $(i,\,j)$ at time $t+\Delta t$ from the known flow field variables at grid points $(i,\,j), (i+1,\,j), (i-1,\,j)$, $(i-1,\,j)$, and $(i,\,j+1)$ at time t. The flow field variables at all the other grid points at time $t+\Delta t$ are obtained in a similar fashion. Once the velocity and temperature fields are obtained at a given time, the local coefficients of the friction and local Nusselt number are calculated from Eqs. (13) and (14). The above numerical technique was used recently by Duwairi and Chamkha[22] and Duwairi et al.[23] for solving the heat transfer problems from vertical surfaces for micro-polar fluids and for the water natural convection heat transfer using a none Boussinesq approximation. An iterative approach is used for both time and the boundary layer thickness in the y-direction, and the incremental steps used in the solution are $\Delta t=0.01$, $\Delta x=0.05$, and $\Delta y=0.05$. The velocity components and temperatures are obtained by the iteration in the governing equation until the convergence is obtained. Here, 10^{-5} is used as a criterion for the convergence for both spatial and time derivatives. At a certain time location, the incremental Δy is first used for the convergence, and later Δt is used insteadly. The solutions presented here are those do not change with different incrementals $\Delta x=0.05$ and $\Delta y=0.05$. In order to verify the accuracy of the present method, a comparison of ours with the similarity solutions obtained by Oosthuizen and Naylor[24] for the steady laminar free convection over a vertical isothermal impermeable plate of Newtonian fluids is performed as shown in Table 1. It is clear from Table 1 that the two results are in excellent agreement.

Table 1 Steady state heat transfer coefficient $h(x,0,\,\infty)$ along the stream wise direction with k_l^*=0, Pr=7, t=∞

x	0.1	0.2	0.4	0.6	0.8	1.0
Present	10.048 21	8.446 53	7.100 78	6.458 97	6.000 24	5.700 23
Ref. [24]	10.040 88	8.443 34	7.099 97	6.415 55	5.970 34	5.646 40

The viscoelsatic fluid effects on this problem are found to be proportional to the dimensionless viscoelastic parameter. The dimensionless viscoelastic parameter $k_l^*=\frac{k_0Gr^{\frac{1}{2}}}{L^2}$ is found to be proportionally direct to the elasticity of the fluid and Grashof numbers. The viscoelastic parameter affects increase as the value of Gr or the buoyancy effect increases for the transient free convection heat transfer problem under consideration. Figure 2 shows the steady state velocity profiles $u(x,y,\,t)$ against the boundary layer thicknesses for different values of modified viscoelastic parameters $k_l^*=0$, 0.6, 1, $Pr=7$, and $x=0.5$. The figure shows that, as the modified viscoelastic parameter increase, the velocity also increases inside the boundary layer since the elastic stresses developed in the fluid are tensile stresses in nature and tend to accelerate the fluid in the flow direction. It is also clear that the hydrodynamic boundary layer thicknesses increases with the increase in the modified viscoelastic parameter. The selected values of the Prandtl number and viscoelsatic parameter lie within acceptable experimental calculated values for viscoelsatic fluids.

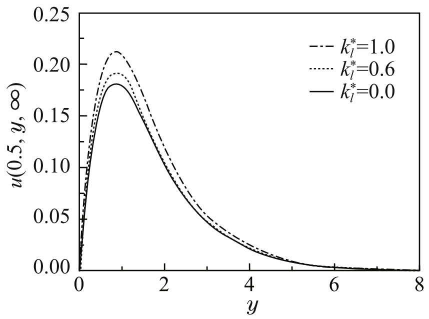

Fig. 2 Steady state velocity profiles for Pr=7 and at x=0.5 (midpoint)

Figure 3 shows the steady state temperature profiles $\Theta(x, y, t)$ against boundary layer thicknesses for different values of viscoelastic parameters $k_l^* = 0$, 0.6, 1, $Pr = 7$, and $x = 0.5$. When the viscoelastic effects increases, the temperatures inside the boundary layer decrease and the temperature gradients near the vertical surface also increase. This is accompanied by the heat transfer enhancements.

Figure 4 shows representative the transient coefficient of friction $C_f Gr^{\frac{3}{4}}$ for different values of the viscoelastic parameters $k_l^* = 0$, 0.6, 1, $Pr = 7$, and $x = 0.5$ and $k_l^* = 0$, 0.4, 0.8, $Pr = 7$, and $x = 0.5$, respectively. It should be noted that the case where $k_l^* = 0$ corresponds to Newtonian fluids. Increasing the viscoelastic parameter has the tendency to increase the local coefficient of friction because of the higher velocities due to the favorable tensile stresses between the fluid layers.

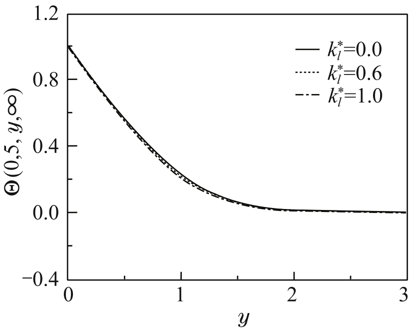

Fig. 3 Steady state temperature profiles for Pr=7 at x=0.5 (midpoint)

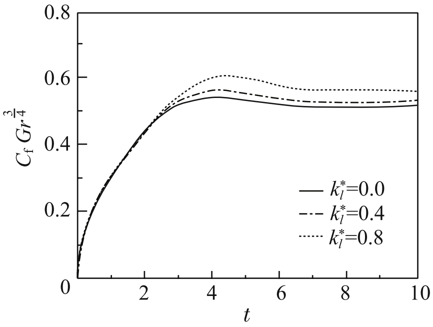

Fig. 4 Transient friction coefficient $C_f Gr^{\frac{3}{4}}$ for Pr=7 at x=0.5 (midpoint)

Figure 5 shows the steady state local coefficients of friction $C_f Gr^{\frac{3}{4}}$ and Nusselt numbers $NuGr^{-\frac{1}{4}}$ against different viscoelastic parameters k_l^* at $x = 0.5$ and $Pr = 7$. The increase of viscoelastic effects increases both the local coefficient of friction and the local Nusselt numbers in spite of small viscoelsatic effects on velocity profiles because the coefficient of friction is a function of the combination of axial and vertical velocity gradients near the surface and of the viscoelsatic parameter.

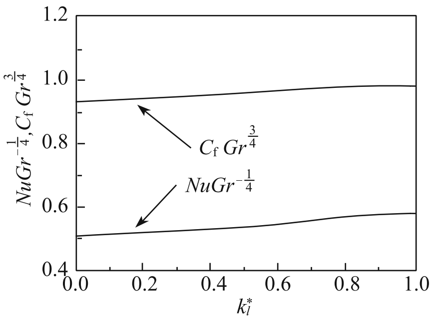

Fig. 5 Steady state local coefficients of friction $C_f Gr^{\frac{3}{4}}$ and local Nusselt number $NuGr^{-\frac{1}{4}}$ via k_l^* for Pr=7 at x=0.5 (midpoint)

4 Conclusions

The transient laminar free convection heat transfer effects from a vertical surface for a viscoelastic fluid are studied. The governing equations are first written in a dimensionless form using a set of variables, and then solved using an explicit finite-difference technique. It is found that the increase of the viscoelsatic parameter increases the velocities inside the boundary layers and consequently increases the coefficient of friction. It is also found that the increase of the viscoelsatic parameter decreases the temperatures inside the boundary layers and consequently increases the coefficient of heat transfer.

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