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ANALYTICAL SOLUTIONS FOR TRANSIENT NATURAL CONVECTION FLOW OF A PARTICULATE SUSPENSION THROUGH A CIRCULAR PIPE

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ABSTRACT

A continuum model for two-phase (fluid/particle) flow induced by natural convection is developed and applied to the problem of transient natural convention flow of a particulate suspension through infinitely long circular pipes. The wall of the pipe is maintained at constant temperatures. Closed-form transient solutions are obtained. A parametric study of physical parameters involved in the problem are done to illustrate the influence of these parameters on the flow and heat transfer aspects of the problem.

Keywods: Two-phase flow, particulate suspension, natural convection, circular pipe, analytical solution.

LIST OF SYMBOLS

C	Fluid-phase specific heat at constant pressure
c_p	Particle-phase specific heat at constant pressure
g	Gravitational acceleration
Gr	Grashof number
H	Dimensionless buoyancy parameter
k	Fluid-phase thermal conductivity
N	Interphase momentum transfer coefficient
N_T	Interphase heat transfer coefficient
P	Fluid-phase hydrostatic pressure
Pr	Fluid-phase Prandtl number
R	Pipe radius
t	Time

T	Fluid-phase temperature
T_o	Fluid-phase temperature at a reference point " o "
T_p	Particle-phase temperature
T_w	Wall temperature
u	Fluid-phase dimensionless velocity
u_p	Particle-phase dimensionless velocity
U	Fluid-phase velocity
U_p	Particle-phase velocity
x , r	Polar coordinates

Greek Symbols

α Velocity inverse Stokes number
β Viscosity ratio
γ Specific heat ratio
ε Temperature inverse Stokes number
η Dimensionless y-coordinate
θ Dimensionless fluid-phase temperature
κ Particle loading
μ Fluid-phase dynamic viscosity
μ_p Particle-phase dynamic viscosity
ρ Fluid-phase density
ρ_p Particle-phase density
τ Dimensionless time

INTRODUCTION

Two-phase (fluid-particle) natural convection flow represents one of the most interesting and challenging areas of research in heat transfer. Such flows are found in a wide range of applications including processes in the chemical and food industries, solar collectors where a particulate suspension is used to enhance absorption of radiation, cooling of electronic equipments, cooling of nuclear reactors and many others. In general, all applications of single-phase flow are valid for two-phase particulate suspension flow because the nature of the real life dictates the presence of contaminating particles in fluids. In spite of this fact, the open literature is full of studies on natural convection flows for single phase and very little work is done on natural convection for two-phase particulate suspensions.

The general area of natural or free convection for a single phase has received a great deal of attention in recent years due to the fact that many applications involve natural convection. The book by Gebhart et al. [1] represents a good source of information on many investigations dealing with the subject. Bhargava and Agarwal [2] have studied fully developed free convection flow in a circular pipe. Gupta et al. [3] have considered laminar free convection flow with and without heat sources through co-axial circular pipes. Barletta [4] have solved the problem of combined forced and free convection with viscous dissipation in a vertical circular duct. Mohammed and Salman [5] have studied natural convection from a uniformly heated vertical circular pipe with different entry restriction configurations.

The above references have dealt with steady and transient natural convection flow of a clean fluid. However, as mentioned above, in real situations, it is hardly possible to find a totally clean dust-free fluid. In many applications the dust particles may be added deliberately or may be present naturally. Depending on the level of particle contamination, the presence of solid particles in fluids has proven to alter the heat transfer characteristics significantly. This has a direct effect on the efficiencies and operation of devices and systems. Therefore, it is of great interest to study natural convection in a two-phase fluid-particle flow.

Forced convection two-phase flows have been considered by many previous investigators. For example, Ritter [6] and Ritter and Peddieson [7] have reported transient two-phase fluid-particle flows in channels and circular pipes and found that the presence of particles caused significant reductions in the flow rates of both phases. Chamkha [8] has reported analytical solutions for transient hydromagnetic two-phase flow in channels and pipes for different pressure gradients. Chamkha [8] confirmed the results of Ritter [6] and Ritter and Peddieson [7] and concluded that the presence of a transverse magnetic field normal to the flow direction caused a retardation effect on the motion of the suspension. Dube and Sharma [9] have also reported solutions for unsteady dusty-gas flow in a circular pipe. Gadiraju et al. [10] have investigated steady two-phase vertical flow in a pipe. Jean and Peddieson [11] have mathematically modelled particulate suspesion flows in vertical circular pipes.

On the other hand, very little work have been reported on natural convection flow of a particlulate suspension over and through different geometries. Chamkha and Ramadan [12] and Ramadan and Chamkha [13] have reported some analytical and numerical results for natural convection flow of a two-phase particulate suspension over an infinite vertical plate. They found that increases in either of the particle loading or the wall particulate slip coefficient caused reductions in the velocities of both phases. Also, Okada and Suzuki [14] have considered buoyancy-induced flow of a two-phase suspension in an enclosure. Al-Subaie and Chamkha [15] have reported analytical solutions for steady natural convection flow of a particulate suspension through a circular pipe. Also, Al-Subaie and Chamkha [16] have studied transient natural convection flow of a particulate suspension through a vertical channel and reported closed-form solutions. However, the present authors were unable to locate related theoretical or experimental work in the literature dealing with transient natural convection laminar flow of a particulate suspension in vertical pipes. Thus, there is a definite need for investigation of such a problem since it is almost impossible to find non-contaminated fluid in real applications. Due to the compexity involved in solving a real two-phase particulate suspension, some assumptions will be made to obtain analytical solutions for transient natural convection laminar flow of particulate suspensions through an infinitely-long vertical pipe.

GOVERNING EQUATIONS

Investigation of two-phase particle/fluid flow in circular pipes requires the start from the equations of basic principles. These are the fluid-phase continuity equation, fluid-phase balance of linear momentum equation, fluid-phase balance of energy equation, particle-phase continuity equation, particle-phase balance of linear momentum equation and particle-phase balance of energy equation. The basic equations can be written assuming small volume fraction (see for instance, Marble [17] and Drew [18]) in the following vector form as:

$$\partial_t \rho + \nabla . (\rho \mathbf{V}) = 0 \tag{1}$$

$$\rho (\partial_t \mathbf{V} + \mathbf{V}.\nabla \mathbf{V}) = -\nabla P + \nabla . (\mu \nabla \mathbf{V}) - \rho_p N (\mathbf{V} - \mathbf{V_p}) + \rho \mathbf{g} \tag{2}$$

$$\rho c (\partial_t T + \mathbf{V}.\nabla T) = \nabla . (k\nabla T) + \rho_p c_p N_T (T_p - T) \tag{3}$$

$$\partial_t \rho_p + \nabla . (\rho_p \mathbf{V_p}) = 0 \tag{4}$$

$$\rho_p (\partial_t \mathbf{V_p} + \mathbf{V_p} .\nabla \mathbf{V_p}) = \nabla . (\mu_p \nabla \mathbf{V_p}) + \rho_p N (\mathbf{V} - \mathbf{V_p}) + \rho_p \mathbf{g} \tag{5}$$

$$\rho_p c_p (\partial_t T_p + \mathbf{V}.\nabla T_p) = - \rho_p c_p N_T (T_p - T) \tag{6}$$

where $\mathbf{V}$ and $\mathbf{V_p}$ are the velocity of fluid and particle phases, T and T_p are the temperatures of fluid and particle phases, $\mathbf{g}$ is the gravity vector, ∇P is the pressure gradient, and T_o is the temperature at a reference point " o " in the pipe. The other parameters, namely ρ , μ , c and k are the density, dynamic viscosity, specific heat and thermal conductivity of the fluid phase, while ρ_p , μ_p and c_p are the particle phase density, dynamic viscosity and specific heat. It should be mentioned that, in the above equations, the terms containing the momentum transfer coefficient N represent the interphase drag between the phases while the terms containing the heat transfer coefficient N_T represent the interphase heat transfer between the fluid and the particle phases. It should be noted that any bold property represents a vector property and ∂_t represents partial differentiation with respect to time. The above governing equations are written in a general form.

Consider transient, laminar, natural convection fully-developed two-phase (fluid-particle) flow through an infinitely-long vertical circular pipe. The fluid phase is assumed to be incompressible, viscous, and Newtonian while the particle phase is assumed to be made up of small spherical solid non-deformable particles of one size and uniform density. The particle phase is assumed to be somewhat dense so that light particle-particle interaction exists. The particle-phase viscous stresses can be used to model particle-particle interaction (see for example, Soo [19], Grace [20], Gidaspow [21] and Sinclair and Jackson [22]). They can be thought of as a natural consequence of the averaging processes employed to model a discrete system of particles as a continuum (see, for instance, Drew and Segal [23], Drew [18]). Since the circular pipe is assumed to be infinitely long, the dependence of variables on the x-direction (axial direction) will be negligible compared with that of the r-direction (radial direction) (see Figure 1). Therefore, all dependent variables (except P) in Equations (1) through (6) will only be functions of r and t. Under these assumptions, the governing equations reduce to:

$$\partial_t U = - \partial_x P + \mu [\partial_{rr} U + (1/r)\partial_r U] - \rho_p N (U - U_p) + \rho g \tag{7}$$

$$\rho c \partial_t T = k [\partial_{rr} T + (1/r)\partial_r T] + \rho_p c_p N_T (T_p - T) \tag{8}$$

$$\rho_p \, \partial_t U_p = \mu_p \, [\, \partial_{rr} U_p + (1/r)\partial_r U_p \,] + \rho_p N \, (U - U_p) - \rho_p \, g \tag{9}$$

$$\rho_p \, c_p \, \partial_t T_p = - \rho_p \, c_p \, N_T \, (T_p - T) \tag{10}$$

It should be noted that the continuity equations of both phases are identically satisfied.

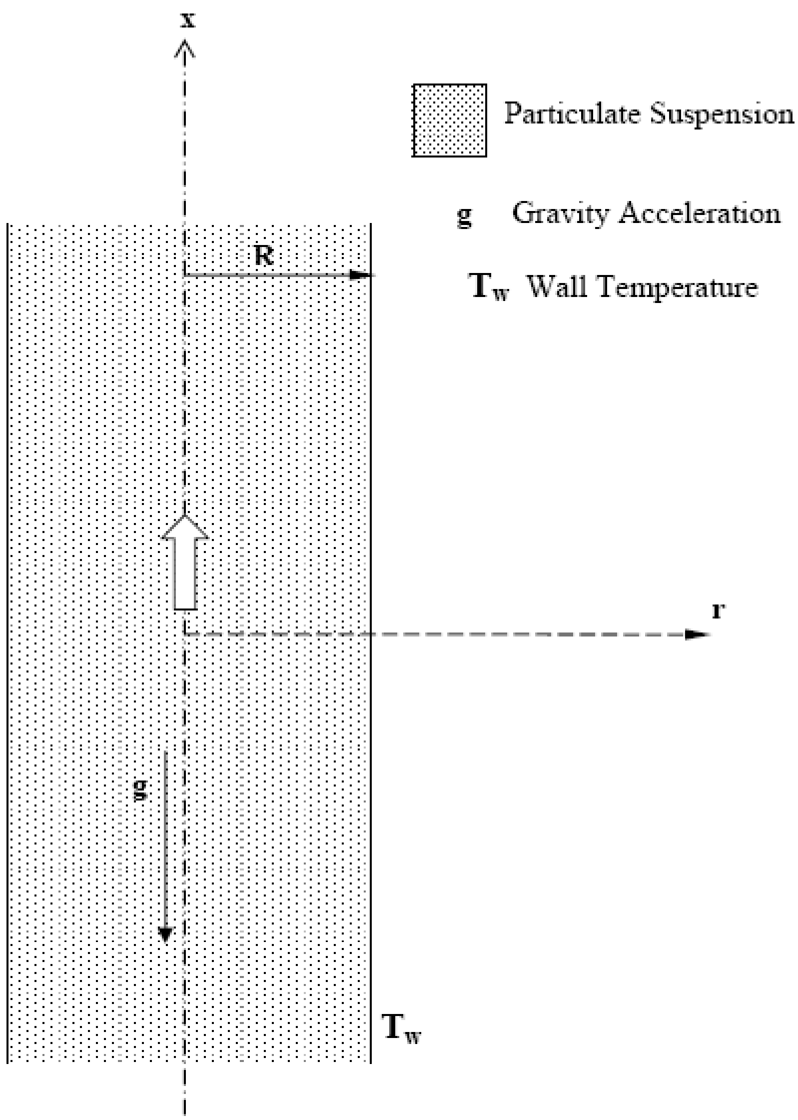

Figure 1. Problem Definition.

The pressure gradient can be eliminated from the linear momentum equation of the fluid phase by evaluating the governing equations at a reference point within the pipe. Let " o " be a reference point within the pipe such that $U = 0$, $T = T_o$, $\rho = \rho_o$, $\mu = \mu_o$, $U_p = U_{po}$, $T_p = T_{po}$, $\rho_p = \rho_{po}$ and $\mu_p = \mu_{po}$. Evaluating the governing equations at this reference point and employing the Boussinesq approximation gives:

$$\partial_t U = \rho_{po}/\rho_o\, g + \mu_o/\rho_o [\, \partial_{rr} U + (1/r)\, \partial_r U \,] - \rho_{po}/\rho_o\, N\, (U - U_p) + \beta^* g\, (T - T_o) \quad (11)$$

where β^* is the volumetric expansion coefficient. The linear momentum equation of the fluid phase, Equation (7), will be replaced now by Equation (11) in the governing equations.

Initially, the fluid phase is assumed to be at rest while the particle phase is assumed to be falling freely under gravity effect. Also, the system is assumed to be at the reference temperature T_o. This can be expressed by

$$U(r,0) = 0\,, T(r,0) = T_o\,, U_p(r,0) = -\, g/N\,, T_p(r,0) = T_o \quad (12)$$

The physical boundary conditions for this problem are:

$$\partial_r U(0,t) = U(R,t) = 0\,, \partial_r T(0,t) = 0\,, T(R,t) = T_w\,,$$
$$\partial_r U_p(0,t) = 0\,, U_p(R,t) = -\, g/N \quad (13a\text{-}f)$$

where R is the pipe radius and T_w is the pipe wall temperature at r = R. Equations (13a) and (13b) indicate a symmetry condition and a no slip condition for the fluid phase in the pipe. Equation (13c) indicates a temperature symmetry condition and Equation (13d) indicates that the fluid temperatures at the wall of the pipe is some constant value T_w. Equations (13e) and (13f) express symmetry and wall boundary conditions for the particle phase in the pipe.

The formulation of the initial-value problem of an infinite vertical pipe is now completed. It is now convenient to non-dimensionalize the governing equations and conditions. This can be accomplished by using the following parameters:

$$r = R\,\eta\,, t = (R^2 \rho/\mu)\,\tau\,, U = (\mu/\rho R)\, u\,, U_p = (\mu/\rho R)\, u_p\,, T = (T_w - T_o)\,\theta + T_o\,,$$
$$T_p = (T_w - T_o)\,\theta_p + T_o \quad (14)$$

where η is the dimensionless transverse coordinate, τ is the dimensionless time, u and u_p are the dimensionless fluid- and particle-phase velocities, respectively, and θ and θ_p are the dimensionless fluid- and particle-phase temperatures, respectively. After performing the mathematical operations, the resulting dimensionless governing equations and conditions can be written as:

$$\partial_\tau u - \partial_{\eta\eta} u - (1/\eta)\, \partial_\eta u + \alpha\,\kappa\, (u - u_p) - Gr\,\theta - \kappa\, H = 0 \quad (15)$$
$$\partial_\tau \theta - (1/Pr)[\, \partial_{\eta\eta}\theta + (1/\eta)\, \partial_\eta \theta \,] - \kappa\,\gamma\,\varepsilon\, (\theta_p - \theta) = 0 \quad (16)$$
$$\partial_\tau u_p - \beta\, [\, \partial_{\eta\eta} u_p + (1/\eta)\, \partial_\eta u_p \,] - \alpha\, (u - u_p) + H = 0 \quad (17)$$
$$\partial_\tau \theta_p + \varepsilon\, (\theta_p - \theta) = 0 \quad (18)$$

where $\alpha = R^2 N\rho/\mu$, $\kappa = \rho_p/\rho$, $Gr = g\,\beta^* R^3 \rho\,(T_w - T_o)/\mu$, $H = gR^3\rho^2/\mu^2$, $\beta = \mu_p/\mu$, $Pr = \mu\, c/k$, $\gamma = c_p/c$ and $\varepsilon = \rho\, N_T R^2/\mu$ are the momentum inverse Stokes number, the particle loading, the Grashof number, buoyancy parameter, the viscosity ratio, the Prandtl number, the specific heat ratio, and the temperature inverse Stokes number.

The dimensionless initial and boundary conditions are:

$$u(\eta,0)=0,\ u_p(\eta,0)=-H/\alpha,\ \theta(\eta,0)=0,\ \theta_p(\eta,0)=0 \quad (19)$$
$$\partial_\eta u(0,\tau)=0,\ u(1,\tau)=0,\ \partial_\eta\theta(0,\tau)=0,\ \theta(1,\tau)=1 \quad (20)$$
$$\partial_\eta u_p(0,\tau)=0,\ u_p(1,\tau)=-H/\alpha \quad (21)$$

It should be mentioned that when $\beta = 0$ (inviscid particle phase), Equations (21) are ignored.

RESULTS AND DISCUSSION

In order to solve this transient problem, it is convenient for simplicity to break the governing equations into two sets of equations by assigning the energy equations of both phases to be the first set, while the momentum equations of both phases to be the second set. This is because the energy equations are uncoupled from the momentum equations. The set of energy equations can be written as:

$$\partial_\tau\theta - (1/Pr)[\partial_{\eta\eta}\theta + 1/\eta\, \partial_\eta\theta] - \kappa\gamma\varepsilon(\theta_p - \theta) = 0 \quad (22)$$
$$\partial_\tau\theta_p + \varepsilon(\theta_p - \theta) = 0 \quad (23)$$

The required initial and boundary conditions can be recorded as:

$$\theta(\eta,0)=0,\ \theta_p(\eta,0)=0 \quad (24)$$
$$\partial_\eta\theta(0,\tau)=0,\ \theta(1,\tau)=1 \quad (25)$$

The solution of the above partial differential Equations (22) and (23) can be expressed in closed form in terms of Fourier-Bessel series that satisfy both the initial and boundary conditions. For convenience, these solutions can be written as

$$\theta(\eta,\tau) = 1 + \sum_{n=1} F_n(\tau)\, J_o(\lambda_n\eta) \quad (26)$$

$$\theta_p(\eta,\tau) = 1 + \sum_{n=1} F_{pn}(\tau)\, J_o(\lambda_n\eta) \quad (27)$$

where $F_n(\tau)$ and $F_{pn}(\tau)$ are the Fourier coefficients and our goal is to find them.

Introducing Equations (26) and (27) into Equations (22) and (23) obtains

$$\sum_{n=1} \{F_n + B_1 F_n - \kappa\gamma\varepsilon F_{pn}\}\, J_o(\lambda_n\eta) = 0 \quad (28)$$

$$\sum_{n=1} \{F_{pn} + \varepsilon F_{pn} - \varepsilon F_n\}\, J_o(\lambda_n\eta) = 0 \quad (29)$$

where a dot represents ordinary differentiation with respect to τ and

$$B_1 = \lambda_n^2/Pr + \kappa\gamma\varepsilon \quad (30)$$

Multiplying each side of Equations (28) and (29) by $\eta\, J_o\,(\lambda_m \eta)$, m is an arbitrary integer , and integrating from 0 to 1 leads to the following equations :

$$F_n + B_1 F_n - \kappa \gamma \varepsilon F_{pn} = 0 \qquad (31)$$
$$F_{pn} + \varepsilon\, F_{pn} - \varepsilon F_n = 0 \qquad (32)$$

Equations (31) and (32) need initial conditions in order to find their solutions. The initial conditions can be obtained by inserting $\tau = 0$ into Equations (26) and (27) and taking into account Equations (24) then integrating with respect to η from 0 to 1 after multiplying each side by $\eta\, J_o\,(\lambda_m \eta)$. Performing the required calculation, one obtains

$$F_n(0) = F_{pn}(0) = -2 / [\lambda_n J_1(\lambda_n)] \qquad (33)$$

Equations (31) and (32) are coupled differential equations. The Laplace transform is a suitable method for solving such equations. Applying the Laplace transform to the mentioned equations subject to Equations (33) yields

$$(s + B_1)\,\mathcal{L}\{F_n\} - \kappa \gamma \varepsilon\, \mathcal{L}\{F_{pn}\} = -2 / [\lambda_n J_1(\lambda_n)] \qquad (34)$$
$$\varepsilon\, \mathcal{L}\{F_n\} - (s + \varepsilon)\, \mathcal{L}\{F_{pn}\} = 2 / [\lambda_n J_1(\lambda_n)] \qquad (35)$$

Solving Equations (34) and (35) for $\mathcal{L}\{F_{pn}\}$, then performing the partial fraction decomposition to the result, gives

$$\mathcal{L}\{F_{pn}\} = B_2 / (s - r_1) + B_3 / (s - r_2) \qquad (36)$$

where

$$r_1 = [\, -B_1 - \varepsilon + \sqrt{(B_1 - \varepsilon)^2 + 4 \kappa \gamma \varepsilon^2}\,] / 2 \qquad (37)$$
$$r_2 = [\, -B_1 - \varepsilon - \sqrt{(B_1 - \varepsilon)^2 + 4 \kappa \gamma \varepsilon^2}\,] / 2 \qquad (38)$$
$$B_2 = \{ -2\,(r_1 + B_1 + \varepsilon) / (r_1 - r_2) \} / [\lambda_n J_1(\lambda_n)] \qquad (39)$$
$$B_3 = \{ -2\,(r_2 + B_1 + \varepsilon) / (r_2 - r_1) \} / [\lambda_n J_1(\lambda_n)] \qquad (40)$$

Now, the Fourier coefficient $F_{pn}(\tau)$ can be determined by inverting the Laplace transform of Equation (36). Doing this gives

$$F_{pn}(\tau) = B_2\, \mathbf{e}^{r_1 \tau} + B_3\, \mathbf{e}^{r_2 \tau} \qquad (41)$$

On the other hand, the Fourier coefficient $F_n(\tau)$ can be easily determined by substituting Equation (41) into (32). Carrying out this substitution and rearranging the result, yields

$$F_n(\tau) = B_5\, \mathbf{e}^{r_1 \tau} + B_6\, \mathbf{e}^{r_2 \tau} \qquad (42)$$

where

$$B_5 = B_2 (r_1 / \varepsilon + 1) \quad (43)$$
$$B_6 = B_3 (r_2 / \varepsilon + 1) \quad (44)$$

The transient fluid- and particle-phase temperature profiles in the pipe can be obtain respectively by substituting the solutions of $F_n (\tau)$ and $F_{pn} (\tau)$ into Equations (26) and (27), respectively. This completes the exact solutions of the set of energy equations.

Now, the set of momentum equations of both phases can be written as

$$\partial_\tau u - \partial_{\eta\eta} u - 1/\eta\, \partial_\eta u + \alpha\, \kappa (u - u_p) - Gr\, \theta - \kappa H = 0 \quad (45)$$
$$\partial_\tau u_p - \beta (\partial_{\eta\eta} u_p + 1/\eta\, \partial_\eta u_p) - \alpha (u - u_p) + H = 0 \quad (46)$$

The corresponding initial and boundary conditions are

$$u (\eta , 0) = 0 , \; u_p (\eta , 0) = - H / \alpha \quad (47)$$
$$\partial_\eta u (0 , \tau) = 0 , \; u (1 , \tau) = 0 , \quad (48)$$
$$\partial_r u_p (0 , \tau) = 0 , \; u_p(1 , \tau) = - H /\alpha \quad (49)$$

Equations (45) and (46) can be solved exactly subject to the mentioned initial and boundary conditions by using assumed solutions of a Fourier-Bessel series form that satisfy both the initial and boundary conditions. For convenience, these assumptions can be written as

$$u (\eta , \tau) = \sum_{n=1} U_n(\tau)\, J_o (\lambda_n \eta) \quad (50)$$

$$u_p (\eta , \tau) = - H / \alpha + \sum_{n=1} U_{pn} (\tau)\, J_o (\lambda_n \eta) \quad (51)$$

where the Fourier coefficients $U_n (\tau)$ and $U_{pn} (\tau)$ are to be determined. Direct substitution of Equations (50) and (51) into Equations (45) and (46) gives

$$\sum_{n=1} \{U_n + B_7 U_n - \kappa\, \alpha\, U_{pn} \}\, J_o (\lambda_n \eta) = Gr\, \theta$$

$$= Gr + Gr \sum_{n=1} F_n\, J_o (\lambda_n \eta) \quad (52)$$

$$\sum_{n=1} \{U_{pn} + B_8 U_{pn} - \alpha\, U_n \}\, J_o (\lambda_n \eta) = 0 \quad (53)$$

where

$$B_7 = \lambda_n^2 + \kappa\, \alpha \quad (54)$$
$$B_8 = \lambda_n^2 \beta + \alpha \quad (55)$$

Multiplying each equation by $\eta\, J_o(\lambda_m\, \eta)$ (to take advantage of the orthogonality property of the Bessel function), and then integrating each equation with respect to η from zero to one yields

$$U_n + B_7\, U_n - \kappa\, \alpha\, U_{pn} = 2\, Gr / [\, \lambda_n\, J_1\, (\, \lambda_n\,)\,] + Gr\, F_n \quad (56)$$
$$U_{pn} + B_8\, U_{pn} - \alpha\, U_n = 0 \quad (57)$$

The initial conditions for Equations (56) and (57) can be obtained, as done in solving the energy equations, by substituting $\tau = 0$ into Equations (50) and (51), multiplying each side by $\eta\, J_o\, (\, \lambda_m\, \eta\,)$ and then integrating with respect to η from zero to one. Doing this gives

$$U_n\, (\, 0\,) = 0 \quad (58)$$
$$U_{pn}\, (\, 0\,) = 0 \quad (59)$$

The Laplace transform can be used to transform the set of linear differential Equations (56) and (57) into a set of algebraic equations as follows:

$$(\, s + B_7\,)\, \mathcal{L}\{U_n\} - \kappa\, \alpha\, \mathcal{L}\{U_{pn}\} = 2\, Gr / [\, \lambda_n\, J_1\, (\, \lambda_n\,)\, s\,] + Gr\, \mathcal{L}\{F_n\} \quad (60)$$
$$\alpha\, \mathcal{L}\{U_n\} - (\, s + B_8\,)\, \mathcal{L}\{U_{pn}\} = 0 \quad (61)$$

Elementary elimination of $\mathcal{L}\{U_n\}$ from the above system of equations gives

$$\mathcal{L}\{U_{pn}\} = 2\alpha\, Gr / [\, \lambda_n\, J_1\, (\, \lambda_n\,)\, s\, (\, s - r_3\,)(\, s - r_4\,)\,] + \alpha\, Gr\, \mathcal{L}\{F_n\} / [\, (\, s - r_3\,)(\, s - r_4\,)\,] \quad (62)$$

where

$$r_3 = [\, - B_7 - B_8 + \surd\, (\, B_8 - B_7\,)^2 + 4\, \kappa\, \alpha^2\,] / 2 \quad (63)$$
$$r_4 = [\, - B_7 - B_8 - \surd\, (\, B_8 - B_7\,)^2 + 4\, \kappa\, \alpha^2\,] / 2 \quad (64)$$

Substituting the expression of $\mathcal{L}\{F_n\}$ from Equation (42) into (62) and applying the partial fractions decomposition technique gives

$$\mathcal{L}\{U_{pn}\} = B_9 / (\, s - r_1\,) + B_{10} / (\, s - r_2\,) + B_{11} / (\, s - r_3\,) + B_{12} / (\, s - r_4\,) + B_{13} / s \quad (65)$$

where

$$B_9 = \alpha\, Gr\, B_5 / [\, (\, r_1 - r_3\,)\, (\, r_1 - r_4\,)\,] \quad (66)$$
$$B_{10} = \alpha\, Gr\, B_6 / [\, (\, r_2 - r_3\,)\, (\, r_2 - r_4\,)\,] \quad (67)$$
$$B_{11} = 2\, \alpha\, Gr / [\, \lambda_n\, J_1\, (\, \lambda_n\,)\, r_3\, (\, r_3 - r_4\,)\,] + \alpha\, Gr\, [\, B_5 / (\, r_3 - r_1\,) + B_6 / (\, r_3 - r_2\,)\,] / (\, r_3 - r_4\,) \quad (68)$$
$$B_{12} = 2\, \alpha\, Gr / [\, \lambda_n\, J_1\, (\, \lambda_n\,)\, r_4\, (\, r_4 - r_3\,)\,]$$

$$+ \alpha \, Gr \, [B_5 / (r_4 - r_1) + B_6 / (r_4 - r_2)] / (r_4 - r_3) \quad (69)$$

$$B_{13} = 2 \, \alpha \, Gr / [r_3 \, r_4 \, \lambda_n \, J_1 (\lambda_n)] \quad (70)$$

Taking the inverse Laplace transform of $\mathcal{L}\{U_{pn}\}$ gives

$$U_{pn} (\tau) = B_9 \, \mathbf{e}^{r_1 \tau} + B_{10} \, \mathbf{e}^{r_2 \tau} + B_{11} \, \mathbf{e}^{r_3 \tau} + B_{12} \, \mathbf{e}^{r_4 \tau} + B_{13} \quad (71)$$

Finally, to find $U_n(\tau)$, it is easier to substitute the above result into Equation (57) and solve it for $U_n (\tau)$. By doing this, one obtains

$$U_n (\tau) = B_{14} \, \mathbf{e}^{r_1 \tau} + B_{15} \, \mathbf{e}^{r_2 \tau} + B_{16} \, \mathbf{e}^{r_3 \tau} + B_{17} \, \mathbf{e}^{r_4 \tau} + B_{18} \quad (72)$$

where

$$B_{14} = B_9 (r_1 + B_8) / \alpha \quad (73)$$
$$B_{15} = B_{10} (r_2 + B_8) / \alpha \quad (74)$$
$$B_{16} = B_{11} (r_3 + B_8) / \alpha \quad (75)$$
$$B_{17} = B_{12} (r_4 + B_8) / \alpha \quad (76)$$
$$B_{18} = B_{13} \, B_8 / \alpha \quad (77)$$

With the solutions of $U_n (\tau)$ and $U_{pn} (\tau)$ known, the solutions for $u (\eta , \tau)$ and $u_p (\eta, \tau)$ can be determined from Equations (50) and (51). This completes the exact solution of the momentum equations of both phases and concludes the results of the transient problem.

Figures 2 and 3 present the evolution of the fluid-phase temperature profile θ and the particle-phase temperature profile θ_p, respectively.

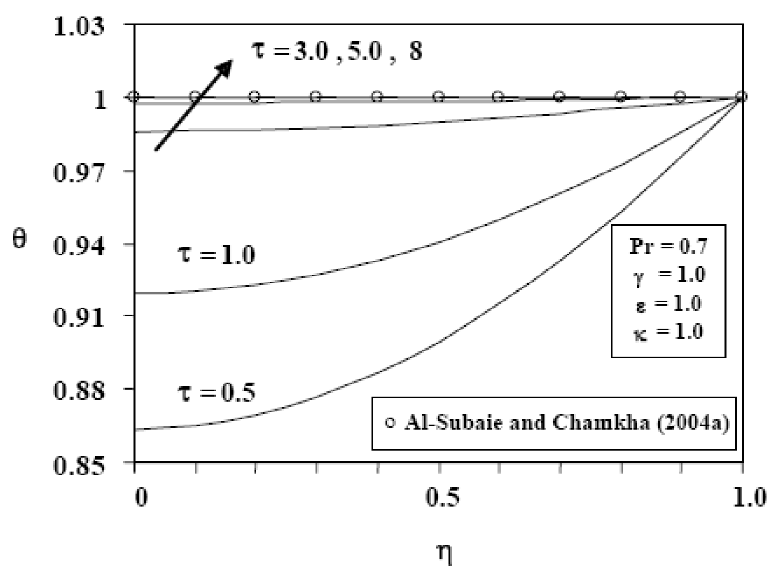

Figure 2. Transient Development of Fluid -PhaseTemperature Profiles.

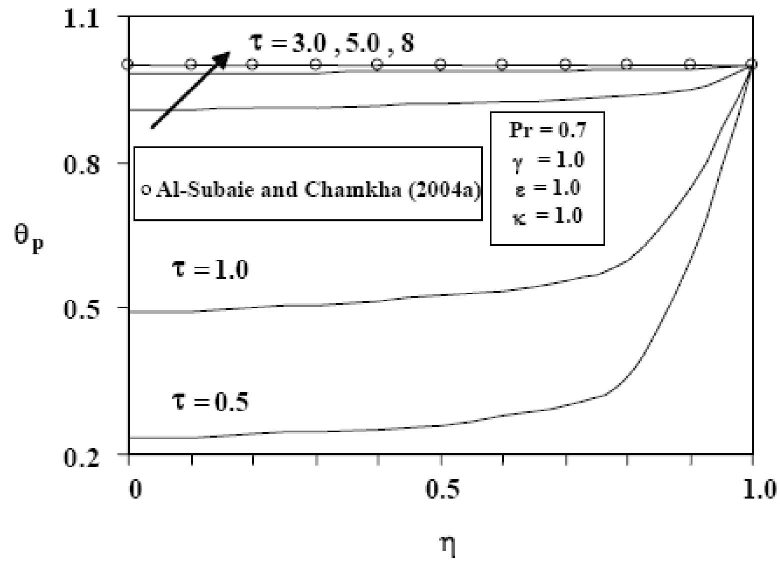

Figure 3. Transient Development of Particle-Ph-ase Temperature Profiles.

The proper transition from transient conditions at small values of the dimensionless time τ to steady conditions at large values of τ is apparent. The steady-state profiles as $\tau \rightarrow \infty$ are consistent with the exact solutions reported earlier by Al-Subaie and Chamkha [15].

Figures 4a,b and 5a,b present the development of the velocity profiles for the fluid and particle phases in the pipe with the dimensionless time τ for both inviscid (β=0) and viscous ($\beta \neq 0$) particle phases, respectively. Again, the proper transition from transient conditions at small values of the dimensionless time τ to steady conditions at large values of τ is apparent. The steady-state profiles at $\tau \rightarrow \infty$ are consistent with the exact solutions reported earlier by Al-Subaie and Chamkha [15].

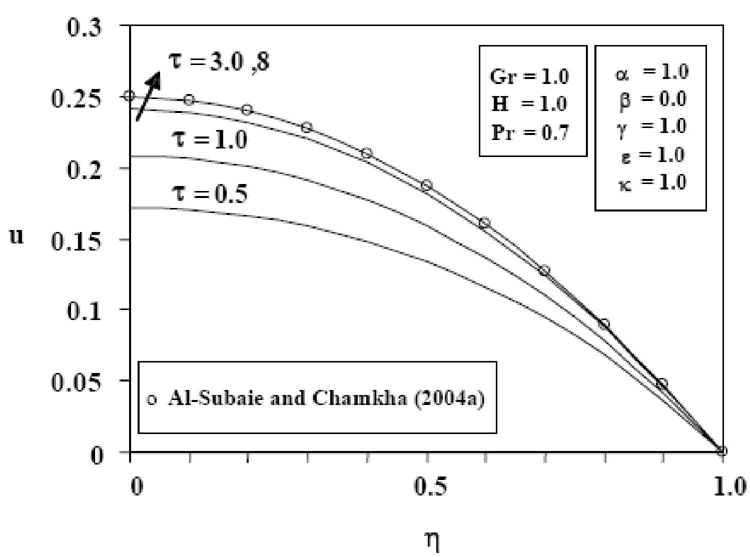

Figure 4a. Transient Development of Fluid -Phase Velocity Profiles at β = 0.

To analyze the effect of the particle loading κ on the solutions developed above, various runs were made to generate data by varying κ at the time $\tau = 1$. The effect of κ on the temperature and velocity profiles of both the fluid and particle phases are shown in Figures 6 through 9.

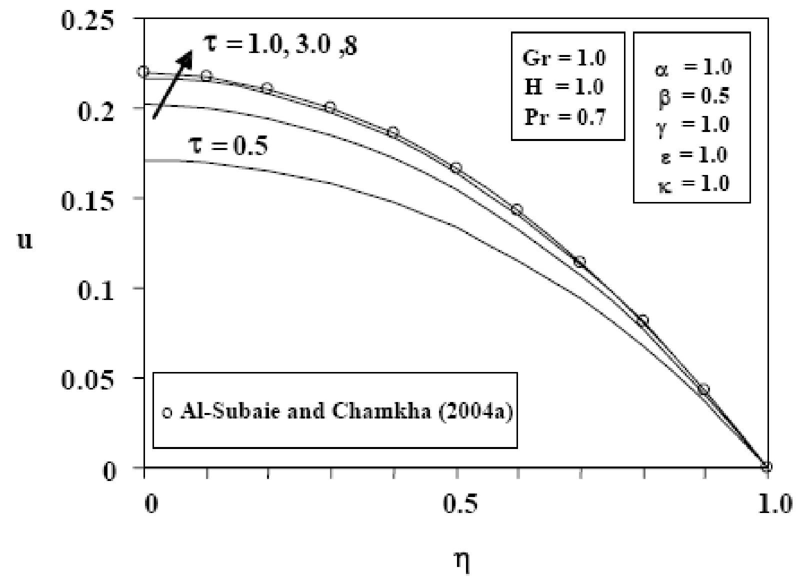

Figure 4b. Transient Development of Fluid -Phase Velocity Profiles at β = 0.5.

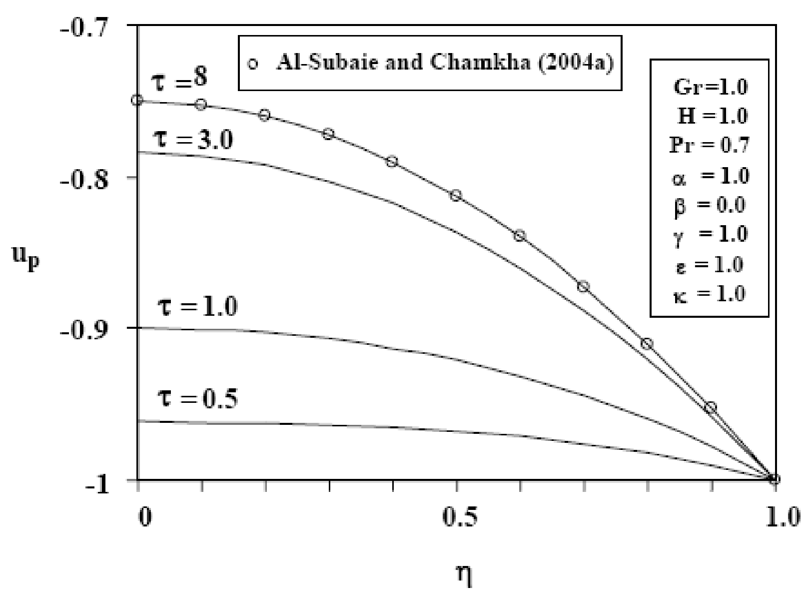

Figure 5a. Transient Development of Particle -Phase Velocity Profiles at β = 0.

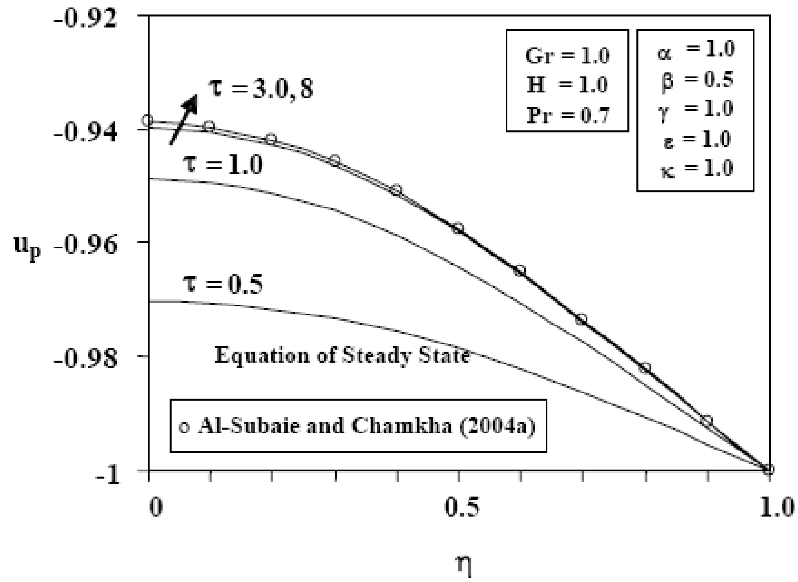

Figure 5b. Transient Development of Particle -Phase Velocity Profiles at β = 0.5.

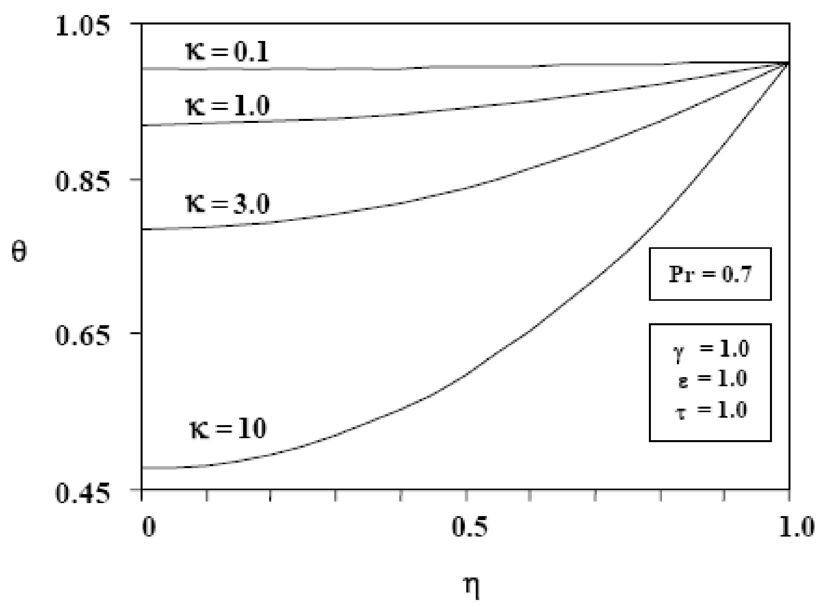

Figure 6. Effects of κ on Fluid-Phase Temperature Profiles at τ = 1.0.

Figures 6 and 7 present the effects of κ at τ = 1 on the fluid-phase temperature θ and the particle-phase temperature θ_p , respectively. As the particle loading increases, more energy exchange between the phases takes place causing the fluid-phase temperature profiles to decrease. Figures 8 and 9 show the effects of κ on the fluid-and particle-phase velocities (u and u_p) at τ = 1, respectively. Increases in the particle loading results in more energy exchange between the phases causing an increase in the magnitude of the viscous or frictional effects for both phases in comparison with the buoyancy effects. This has the direct effect of decreasing the velocities of both phases in the pipe in the transient range as clearly depicted in Figures 8 and 9.

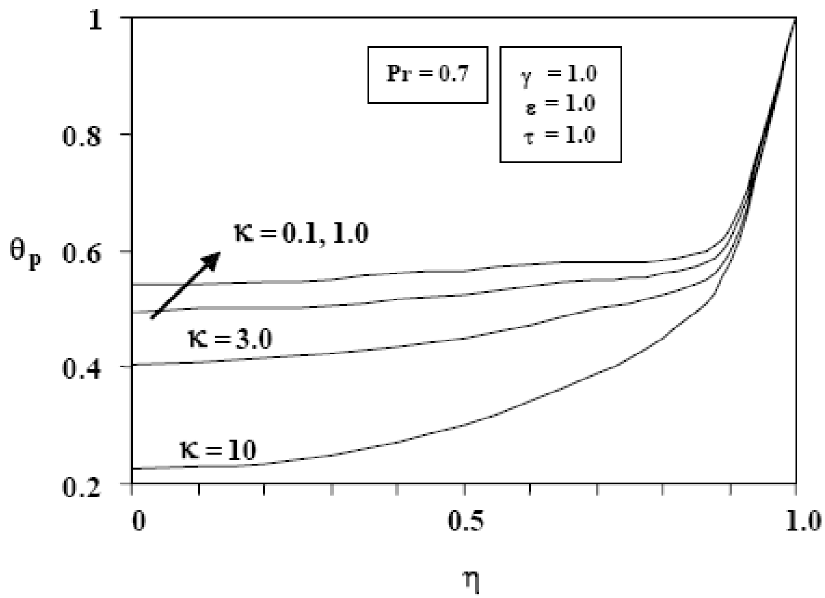

Figure 7. Effects of κ on Particle-Phase Temperature Profiles at τ = 1.0.

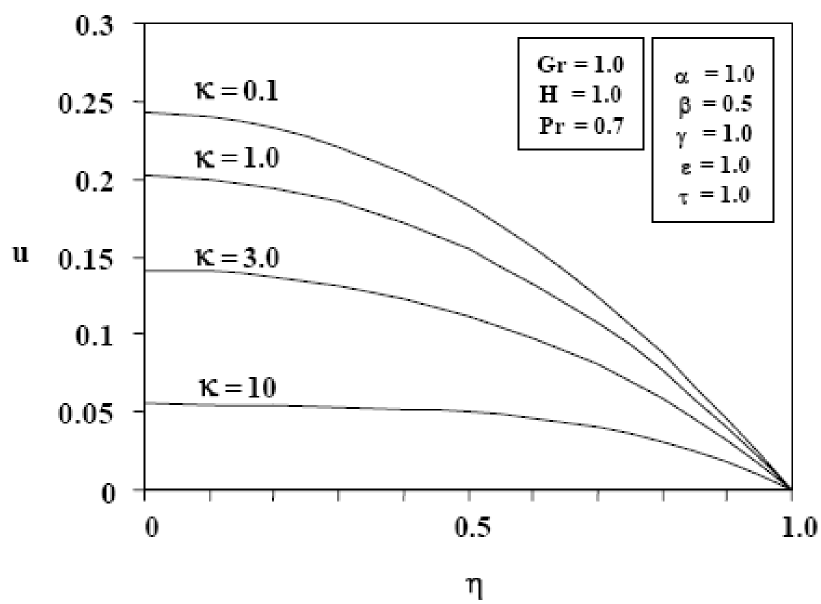

Figure 8. Effects of κ on Fluid-Phase Velocity Profiles at τ = 1.

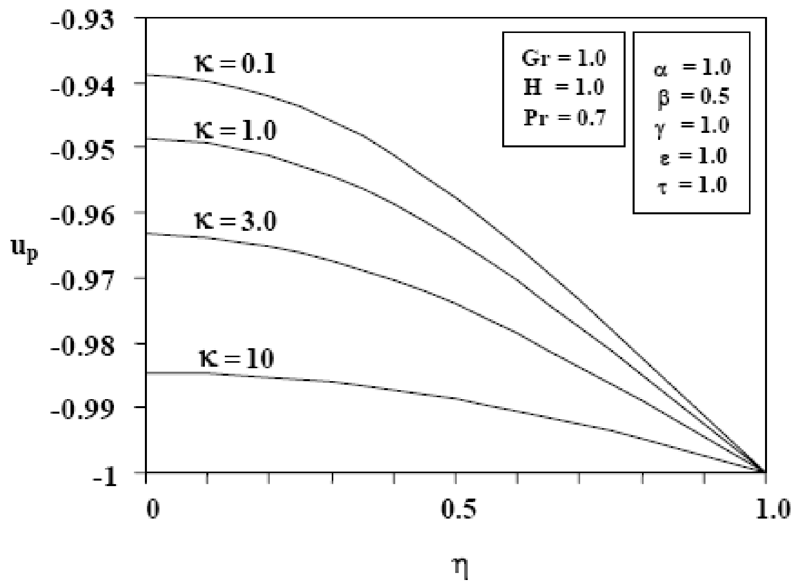

Figure 9. Effects of κ on Particle -Phase Velocity Profiles at τ = 1.

CONCLUSION

The mathematical modelling of natural convection flow of a particulate suspension was formulated in its general form by stating the conservation laws of mass, linear momentum, and energy for both the fluid and particle phases. The general formulation took into account the effects of particle-phase viscosity. The governing equations were non-dimensionlized and solved analytically. Closed-form solutions for the general transient problem were obtained. The proper transition from transient conditions at small values of the dimensionless time τ to steady conditions at large values of τ was predicted. The steady-state profiles as $\tau \to \infty$ were consistent with the exact solutions reported for the steady-state. In the transient range, as the particle loading increased more energy exchange between the phases took place causing the temperature profiles to decrease and the magnitudes of the fluid- and particle-phase velocities to decrease.

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