

## begell house, inc.

Journal Production

50 Cross Highway

Redding, CT 06896

**Phone:** 1-203-938-1300

**Fax:** 1-203-938-1304

**Begell House Production Contact :** [journals@begellhouse.com](mailto:journals@begellhouse.com)

Dear Corresponding Author,

Effective April 2011 Begell House will no longer provide corresponding authors with a print copy of the issue in which their article appears. Corresponding authors will now receive a pdf file of the final version of their article that has been accepted for publication.

Please note that the pdf file provided is for your own personal use and is not to be posted on any websites or distributed in any manner (electronic or print). Please follow all guidelines provided in the copyright agreement that was signed and included with your original manuscript files.

Any questions or concerns pertaining to this matter should be addressed to [journals@begellhouse.com](mailto:journals@begellhouse.com)

Thank you for your contribution to our journal and we look forward to working with you again in the future.

.

Sincerely,

*Michelle Amoroso*

Michelle Amoroso

Production Department

# SOLUTE DISPERSION BETWEEN TWO PARALLEL PLATES CONTAINING POROUS AND FLUID LAYERS

J. Prathap Kumar,<sup>1</sup> J. C. Umavathi,<sup>1</sup> Ali J. Chamkha,<sup>2,\*</sup> & Ashok Basawaraj<sup>1</sup>

<sup>1</sup>Department of Mathematics, Gulbarga University, Gulbarga 585 106, Karnataka, India

<sup>2</sup>Manufacturing Engineering Department, The Public Authority for Applied Education and Training, Shuweikh 70654, Kuwait

\*Address all correspondence to Ali J. Chamkha E-mail: achamkha@yahoo.com

Original Manuscript Submitted: 10/12/2011; Final Draft Received: 2/1/2012

*This paper presents an analytical solution for the dispersion of a solute between two parallel plates consisting of two regions, one region filled with porous matrix and another region filled with purely viscous fluid. The Brinkman model is used to define the flow through the porous matrix. The fluids in both regions of the parallel-plate channel are incompressible, and their transport properties are assumed to be constant. The closed-form solutions are obtained in both fluids regions of the channel. The results are tabulated for various values of porous parameter, viscosity ratio, and pressure gradient on the effective dispersion coefficient and volumetric flow rate. It is found that the effective Taylor dispersion coefficient decreases as the porous parameter increases. The validity of the results obtained for a two-fluid model is compared with the available one-fluid model in the presence of porous matrix, and the values agree very well. The effective dispersion coefficient and volumetric flow rate are also found for a two-fluid model in the absence of porous matrix and are verified with the available one-fluid model, and the results are in good agreement.*

**KEY WORDS:** Taylor dispersion, immiscible fluids, horizontal channel, porous medium

## 1. INTRODUCTION

Flow through and past porous media has attracted considerable interest in recent years because of its importance in science, engineering, and technology (see Kaviany, 1999; Nield and Bejan, 1999; Vafai, 2000). Much interest has also been evinced in (1) the efficient recovery of crude oil from the pores of petroleum reservoir rocks by displacement with immiscible water (Taber, 1980; Posner and Gill, 1973) where a relation between the displacement rate and pressure is desired; (2) the tiny dust particles floating in the air, known as aerosols and which are gradually choking people to death, where an understanding of the spreading of the aerosols is required—the main interest in all of these cases is the study of dispersion (i.e., spreading leading to flow-enhanced diffusion) in porous media; and (3) the dependence of the rate of diffusion on

porosity, which is of considerable importance to the plant physiologists and hence the development of a theory to calculate the diffusion coefficient through porous media needs little justification.

The two important phenomena in the study of dispersion in laminar flow are the molecular diffusion and convection. The flow and inhomogeneity due to the existence of the pore system are responsible for the convective diffusion. The variation of permeability also contributes to the convective diffusion. However, the variation in concentration within the liquid phase gives rise to molecular diffusion and is significant at low velocities. This phenomenon of dispersion in porous media is generally studied by two approaches, namely, the probabilistic approach or the cell model approach and the deterministic approach using the average procedure. The former is due to Simpson (1969) and the latter is due to

### NOMENCLATURE

$C_i$ concentration of the solute $dP_i/dX_i$ pressure gradient $D_i$ molecular diffusion coefficient $D$ ratio of molecular diffusion coefficient ( $D_2/D_1$ ) $F_i$ effective dispersion coefficient $h$ distance between the plates $L$ typical length along the flow direction $m$ ratio of viscosities ( $\mu_2/\mu_1$ ) $n$ density ratio ( $\rho_1/\rho_2$ ) $P_i$ nondimensional pressure gradient $Q_i$ volumetric flow rate	$U_i$ velocity $\bar{u}_i$ nondimensional average velocity  <b>Greek Symbols</b> $\eta$ dimensionless length $\kappa_i$ permeability of the porous medium $\mu_i$ dynamic viscosity $\sigma$ porous parameter $\rho_i$ density of the fluid  <b>Subscripts</b> $i = 1, 2$ where 1, 2 are quantities for region 1 and region 2, respectively
--	--

Bear and Bechmat (1965). The results of the cell model (Simpson, 1969) show the dimensionless diffusion coefficient to be independent of the Reynolds number  $R$  (based on the mean grain diameter) and approaches asymptotically to a constant value as  $R$  increases, whereas the basic idea of the deterministic approach, developed by Bear and Bechmat (1965), is to replace the porous matrix and the liquid filling the void space with their respective fictitious interpenetrating continua, each of which is occupying the entire flow domain. The results of this model show that the dimensionless diffusion coefficient increases with increasing Reynolds number and approaches infinity as  $R \rightarrow \infty$ . This is in contradiction to the cell model results and the experimental data of Harleman et al. (1963). This promoted us to develop a deterministic model that would agree with the experimental and the cell model results. A convenient way to approach this problem is to consider Taylor's (1953) model based on the boundary layer type of flow pattern. In the case of porous media, the boundary layer type of flow pattern can be obtained by Brinkman (1947) model, which incorporates both frictional force offered by the solid particles to the fluid and the viscous shear. The Brinkman equation is of the boundary layer type, and the existence of a boundary layer near the surface of a porous media has been experimentally demonstrated by Beavers and Joseph (1967) and theoretically established by Saffman (1971) and Rudraiah and Veerabhadraiah (1977). Rudraiah (1976) has used

this Brinkman model to find the effect of homogeneous and heterogeneous reactions on the concentration distribution in a fixed bed chemical reactor and has shown that the Brinkman model gives a reasonable result.

The work on dispersion in the presence of porous media in the literature was studied for one fluid model. However, in realistic situations, the fluid system often times consists of two (and possibly more) separate immiscible liquids, a layer of one liquid over layer of another liquid. The problem formulation now contains additional dynamical ingredients such as the interfacial stress and the deformation of the interface shape. Also, a multilayered liquid arrangement provides an improved model for buoyancy-driven convection process in growing high-quality crystals. Derjani et al. (1986) and Nield (1977, 1983) have studied the thermal instabilities of a superposed porous and fluid layer using Darcy's law together with matching conditions. Masuoka (1974) has observed convective flow in a layer of fluid heated from below and divided by a horizontal porous wall. He has found that the porous wall suppresses the convection. Neale and Nader (1974) proposed the use of Brinkman-extended Darcy equation to account for the macroscopic viscous stress in the porous medium. They also suggested that at the interface, the macroscopic viscous shear stress in the porous medium is equal to the shear stress on the fluid side at the interface between the fluid layer and the porous medium. Somerton and Catton (1982) and Catton (1985) have studied the sta-

bility and heat transfer in a superposed horizontal porous and fluid layer using Neale and Nader (1974) matching conditions.

Alzami and Vafai (2001) analyzed fluid flow and heat transfer interfacial conditions between a porous medium and a fluid layer. Kuznetsov (2000) gave a detailed introduction on the applications and analytical studies of forced convection in partly porous configurations. The first exact solution for the fluid flow in the interface was presented in Vafai and Kim (1990). In this study, the shear stress in the fluid and the porous medium were taken to be equal at the interface region. Using the same model, Malashetty et al. (2001), Umavathi et al. (2010), and Prathap Kumar et al. (2009) studied flow and heat transfer of fluid-saturated porous media for two- or three-fluid models.

The literature on hydrodynamic dispersion in porous media is very sparse despite its versatile applications in many branches of science, engineering, and technology. Notable practical problems that require the study of dispersion in porous medium are the extraction of energy from geothermal regions, solar collectors with porous absorbers, drag permeation through human skin, and biomechanical applications such as cartilage in synovial joints (see Sueiu et al., 2003; Ng et al., 2005). This dispersion in porous medium is the macroscopic outcome of the actual movements of the individual solute particles through the pores and the various physical and chemical phenomena that take place within the pores. Recently, Rudraiah and Ng (2007) analyzed the dispersion in porous media considering all possible models for one fluid model.

Based on our review of the current literature, it is evident that very few studies are available on the dispersion in porous medium for a two-fluid model. Looking at the various applications as mentioned earlier, the main objective of this study is to analyze solute dispersion between two parallel plates filled with porous and viscous immiscible fluid layers using Taylor's (1953) model.

## 2. MATHEMATICAL FORMULATION

The physical configuration considered in this study is shown in Fig. 1. Consider the laminar flow of two immiscible fluids between two parallel plates at a distance  $2H$  apart, taking the  $x$  axis along the midsection of the channel and the  $y$  axis perpendicular to the walls. Region 1 ( $-h \leq Y \leq 0$ ) is filled with a fluid-saturated porous medium of density  $\rho_1$  and viscosity  $\mu_1$  under a uniform pressure gradient  $dP_1/dX$  with permeability  $\kappa$ , whereas Region 2 ( $0 \leq Y \leq h$ ) is filled with another viscous fluid

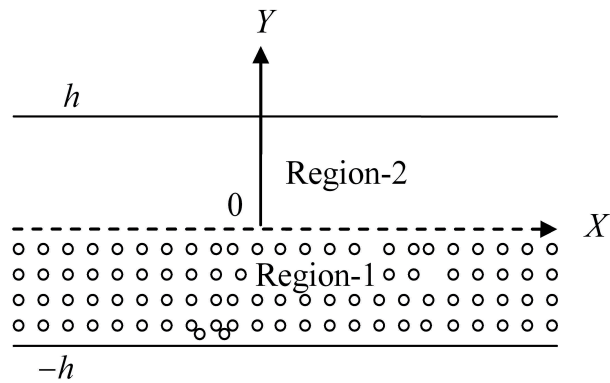


FIG. 1: Physical configuration

of density  $\rho_2$  and viscosity  $\mu_2$  under a uniform pressure gradient  $dP_2/dX$ . The fluids in both the regions are Newtonian fluids. The transport properties of both fluids are assumed to be constant. It is also assumed that the fluids are incompressible and the flow is steady, laminar, and fully developed. The flow in both regions of the parallel-plate channel is assumed to be driven by common constant pressure gradients. Under these assumptions, the governing equations of motion for incompressible fluids are

Region 1:

$$\frac{d^2 U_1}{dY^2} - \frac{1}{\kappa} U_1 = \frac{1}{\mu_1} \frac{dP_1}{dX} \quad (1)$$

Region 2:

$$\frac{d^2 U_2}{dY^2} = \frac{1}{\mu_2} \frac{dP_2}{dX} \quad (2)$$

where  $U_i$  is the  $X$  component of fluid velocity and  $P_i$  is the pressure. The subscripts 1 and 2 denote the values for Region 1 and Region 2, respectively.

The boundary conditions on velocity are the no-slip conditions requiring that the velocity must vanish at the walls. In addition, continuity of velocity and shear stress at the interface is assumed. With these assumptions, the boundary and interface conditions on velocity become

$$\begin{aligned} U_1 &= 0 \text{ at } Y = -h; \quad U_2 = 0 \text{ at } Y = h \\ U_1 &= U_2 \text{ and } \mu_1 \frac{dU_1}{dY} = \mu_2 \frac{dU_2}{dY} \text{ at } Y = 0 \end{aligned} \quad (3)$$

Using the nondimensional parameters

$$\begin{aligned} \eta &= \frac{Y}{h}, \quad u_1 = \frac{\rho_1 h}{\mu_1} U_1, \quad u_2 = \frac{\rho_2 h}{\mu_2} U_2, \quad x = \frac{X}{h}, \\ p_1^* &= \frac{P_1}{\rho_1 (\nu_1/h)^2}, \quad p_2^* = \frac{P_2}{\rho_2 (\nu_2/h)^2}, \quad \sigma = \frac{h}{\sqrt{\kappa}} \end{aligned} \quad (4)$$

the Eqs. (1)–(3) become

$$\frac{d^2 u_1}{d\eta^2} - \sigma^2 u_1 = p_1 \tag{5}$$

$$\frac{d^2 u_2}{d\eta^2} = p_2 \tag{6}$$

$$\begin{aligned} u_1 = 0 \quad \text{at} \quad \eta = -1; \quad u_2 = 0 \quad \text{at} \quad \eta = 1 \\ u_1 = mn u_2 \quad \text{and} \quad \frac{du_1}{d\eta} = m^2 n \frac{du_2}{d\eta} \quad \text{at} \quad \eta = 0 \end{aligned} \tag{7}$$

where

$$p_1 = \frac{dp_1^*}{dx}, \quad p_2 = \frac{dp_2^*}{dx}, \quad m = \mu_2/\mu_1, \quad n = \rho_1/\rho_2.$$

Solutions of Eqs. (5) and (6) are

$$u_1 = a_1 \cosh(\sigma \eta) + a_2 \sinh(\sigma \eta) - \frac{p_1}{\sigma^2} \tag{8}$$

$$u_2 = \frac{p_2 \eta^2}{2} + a_3 \eta + a_4 \tag{9}$$

where  $a_1, a_2, a_3,$  and  $a_4$  are integrating constants that are evaluated using boundary and interface conditions as given in Eq. (7). The numerical values of velocity distribution  $u_1$  and  $u_2$  is tabulated in Table 1.

From Eqs. (8) and (9), the average velocities become

$$\begin{aligned} \bar{u}_1 &= \frac{1}{2} \int_{-1}^0 u_1 d\eta \\ &= \frac{1}{2\sigma} \left( a_1 \sinh(\sigma) + a_2 \cosh(\sigma) - \frac{p_1}{\sigma} \right) \end{aligned} \tag{10}$$

$$\bar{u}_2 = \frac{1}{2} \int_0^1 u_2 d\eta = \frac{1}{2} \left( \frac{p_2}{6} + \frac{a_3}{2} + a_4 \right) \tag{11}$$

We assume that a solute diffuses in the absence of first-order irreversible chemical reaction in the liquid under isothermal conditions. The equation for the concentration  $C_1$  of the solute for the first region satisfies

$$\frac{\partial C_1}{\partial t} + u_1 \frac{\partial C_1}{\partial X} = D_1 \left( \frac{\partial^2 C_1}{\partial X^2} + \frac{\partial^2 C_1}{\partial Y^2} \right) \tag{12}$$

Similarly, the equation for the concentration  $C_2$  of the solute for the second region satisfies

$$\frac{\partial C_2}{\partial t} + u_2 \frac{\partial C_2}{\partial X} = D_2 \left( \frac{\partial^2 C_2}{\partial X^2} + \frac{\partial^2 C_2}{\partial Y^2} \right) \tag{13}$$

in which  $D_1$  and  $D_2$  are the molecular diffusion coefficients (assumed constant) for the first and second regions, respectively. We now assume that the longitudinal diffusion is much less than the transverse diffusion, that is,

$$\frac{\partial^2 C_1}{\partial X^2} \ll \frac{\partial^2 C_1}{\partial Y^2} \quad \frac{\partial^2 C_2}{\partial X^2} \ll \frac{\partial^2 C_2}{\partial Y^2}$$

Furthermore, we note that when the transverse diffusion is not sufficiently great to wipe out altogether the effect of longitudinal convection, the transverse distribution of concentration will not be uniform unless the longitudinal distribution is also uniform. This observation is essential to the solution of the problem for we see that if the longitudinal distribution is uniform, the rate at which the dissolved substance passes a section which moves with the mean speed of flow is zero. On the other hand, if there is a small longitudinal gradient of concentration along the tube convection will give rise to a small transverse variation of concentration, which in turn will give rise to a small transport of the solute across a section which moves with the mean speed. It is evident that this small transport and the small longitudinal concentration gradient must be

**TABLE 1:** Velocity distribution for different values of porous parameter

$\eta$	Two fluid			One fluid		
	$\sigma = 0.01$	$\sigma = 2$	$\sigma = 4$	$\sigma = 0.01$	$\sigma = 2$	$\sigma = 4$
-1	0	0	0	0	0	0
-0.75	0.2187445	0.1084687	0.0432440	0.2187419	0.0936811	0.0394583
-0.5	0.3749904	0.1808113	0.0655729	0.3749852	0.1474614	0.0538895
-0.25	0.4687385	0.2354934	0.0912395	0.4687307	0.1750686	0.0589684
0	0.4999896	0.2864726	0.1481218	0.4999792	0.1835494	0.0602113
0.25	0.4687422	0.3086044	0.2048414	0.4687307	0.1750686	0.0589684
0.5	0.3749948	0.2682363	0.1990609	0.3749852	0.1474614	0.0538895
0.75	0.2187474	0.1653681	0.1307805	0.2187419	0.0936811	0.0394583
1	0	0	0	0	0	0

proportional to one another. Thus the combined effect of longitudinal convection and transverse diffusion is to disperse the tracer longitudinally relative to a frame moving at the mean speed of flow by a mechanism which obeys the same law as ordinary one-dimensional diffusion relative to a fluid at rest. Thus Eqs. (12) and (13) for concentration now take the form for Region 1 and Region 2, respectively, as

$$\frac{\partial C_1}{\partial t} + u_1 \frac{\partial C_1}{\partial Y} = D_1 \frac{\partial^2 C_1}{\partial Y^2} \quad (14)$$

$$\frac{\partial C_2}{\partial t} + u_2 \frac{\partial C_2}{\partial Y} = D_2 \frac{\partial^2 C_2}{\partial Y^2} \quad (15)$$

Since we are considering convection across a plane moving with the mean velocity of the flow, then relative to this plane, the fluid velocity is given by

$$u_{1x} = u_1 - \bar{u} = a_1 \cosh(\sigma\eta) + a_2 \sinh(\sigma\eta) + l_1 \quad (16)$$

for Region 1 and

$$u_{2x} = u_2 - \bar{u} = \frac{p_2\eta^2}{2} + a_3\eta + l_2 \quad (17)$$

for Region 2, and  $\bar{u}$  is the sum of average velocities of Region 1 and Region 2.

Introducing the dimensionless quantities

$$\theta_1 = \frac{t_1}{\bar{t}_1}, \quad \bar{t}_1 = \frac{L}{\bar{u}_1}, \quad \theta_2 = \frac{t_2}{\bar{t}_2}, \quad \bar{t}_2 = \frac{L}{\bar{u}_2},$$

$$\xi_1 = \frac{X - \bar{u}_1 t}{L_1}, \quad \xi_2 = \frac{X - \bar{u}_2 t}{L}, \quad \eta = \frac{Y}{h} \quad (18)$$

Eqs. (14) and (15) become

Region 1:

$$\frac{1}{t} \frac{\partial C_1}{\partial \theta_1} + \frac{u_{1x}}{L} \frac{\partial C_1}{\partial \xi} = \frac{D_1}{h^2} \frac{\partial^2 C_1}{\partial \eta^2} \quad (19)$$

Region 2:

$$\frac{1}{t} \frac{\partial C_2}{\partial \theta_2} + \frac{u_{2x}}{L} \frac{\partial C_2}{\partial \xi} = \frac{D_2}{h^2} \frac{\partial^2 C_2}{\partial \eta^2} \quad (20)$$

where  $L$  is the typical length along the flow direction. If Taylor's (1953) longitudinal condition is valid in the present problem, then the partial equilibrium may be assumed in any cross section of the channel and the variations of  $C_1$  and  $C_2$  with  $\eta$  are calculated from Eqs. (19) and (20) as

Region 1:

$$\frac{\partial^2 C_1}{\partial \eta^2} = \frac{h^2}{D_1 L} u_{1x} \frac{\partial C_1}{\partial \xi_1} \quad (21)$$

Region 2:

$$\frac{\partial^2 C_2}{\partial \eta^2} = \frac{h^2}{D_2 L} u_{2x} \frac{\partial C_2}{\partial \xi_2} \quad (22)$$

To solve these equations, we use the following two types of boundary conditions. The first is connected with insulated type of boundary conditions, namely,

$$\frac{\partial C_1}{\partial \eta} = 0 \text{ at } \eta = -1 \text{ and } \frac{\partial C_2}{\partial \eta} = 0 \text{ at } \eta = 1 \quad (23)$$

which expresses the fact that the walls of the channel are impermeable. However, in many biological problems, the condition at the upper wall is conducting and the lower wall is insulating. In other words,

$$\frac{\partial C_1}{\partial \eta} = 0 \text{ at } \eta = -1 \text{ and } C_2 = 1 \text{ at } \eta = 1 \quad (24)$$

where the former represents the impermeable and the latter the permeable.

## 2.1 Case 1a: Diffusion of a Tracer with Impermeable (Insulating) Wall Conditions

Equations (21) and (22) are solved exactly for  $C_1$  and  $C_2$  using boundary conditions as defined in Eq. (23), which are given by

Region 1:

$$C_1 = \frac{h_1^2}{D_1 L_1} \frac{\partial C_1}{\partial \xi_1} \left( \frac{a_1 \cosh(\sigma\eta)}{\sigma^2} + \frac{a_2 \sinh(\sigma\eta)}{\sigma^2} + \frac{l_1 \eta^2}{2} + b_1 \eta \right) + C_{01} \quad (25)$$

Region 2:

$$C_2 = \frac{h_2^2}{D_2 L_2} \frac{\partial C_2}{\partial \xi_2} \left( \frac{P_2 \eta^4}{24} + \frac{a_3 \eta^3}{6} + \frac{l_2 \eta^2}{2} + b_3 \eta \right) + C_{02} \quad (26)$$

where  $C_{01}$  and  $C_{02}$  being constants to be determined using the entry conditions.

The volumetric flow rates at which the solute is transported across a section of the channel of unit breadth  $Q_1$  (Region 1) and  $Q_2$  (Region 2) using Eqs. (16), (17) and (25), (26), respectively, are given by

$$Q_1 = h \int_{-1}^0 C_1 u_{1x} d\eta = -\frac{h^3}{D_1 L} \frac{\partial C_1}{\partial \xi_1} \int_{-1}^0 C_{11} u_{1x} d\eta \quad (27)$$

where

$$C_{11} = -\left( \frac{a_1 \cosh(\sigma\eta)}{\sigma^2} + \frac{a_2 \sinh(\sigma\eta)}{\sigma^2} + \frac{l_1 \eta^2}{2} + b_1 \eta \right)$$

and

$$Q_2 = h \int_0^1 C_2 u_{2x} d\eta = -\frac{h^3}{D_2 L} \frac{\partial C_2}{\partial \xi_2} \int_0^1 C_{22} u_{2x} d\eta \quad (28)$$

where

$$C_{22} = -\left(\frac{P_2 \eta^4}{24} + \frac{a_3 \eta^3}{6} + \frac{l_2 \eta^2}{2} + b_3 \eta\right)$$

Following Taylor (1953), we assume that the variations of  $C_1$  and  $C_2$  with  $\eta$  are small compared with those in the longitudinal direction, and if  $C_{m1}$  and  $C_{m2}$  are the mean concentration over a section,  $\partial C_1 / \partial \xi_1$  and  $\partial C_2 / \partial \xi_2$  are indistinguishable from  $\partial C_{m1} / \partial \xi_1$  and  $\partial C_{m2} / \partial \xi_2$ , respectively, so that Eqs. (27) and (28) may be written as Region 1:

$$Q_1 = -D_1^* \frac{\partial C_{m1}}{\partial \xi_1} \quad (29)$$

Region 2:

$$Q_2 = -D_2^* \frac{\partial C_{m2}}{\partial \xi_2} \quad (30)$$

The fact that no material is lost in the process is expressed by the continuity equation for  $C_{m1}$  and  $C_{m2}$ , namely,

Region 1:

$$\frac{\partial Q_1}{\partial \xi_1} = -2 \frac{\partial C_{m1}}{\partial t} \quad (31)$$

Region 2:

$$\frac{\partial Q_2}{\partial \xi_2} = -2 \frac{\partial C_{m2}}{\partial t} \quad (32)$$

Equations (31) and (32) using Eqs. (27) and (28) become

Region 1:

$$\frac{\partial C_{m1}}{\partial t} = \frac{D_1^*}{2} \frac{\partial^2 C_{m1}}{\partial \xi_1^2} \quad (33)$$

Region 2:

$$\frac{\partial C_{m2}}{\partial t} = \frac{D_2^*}{2} \frac{\partial^2 C_{m2}}{\partial \xi_2^2} \quad (34)$$

which are the equations governing the longitudinal dispersion, where

$$D_1^* = \frac{h^2}{2D_1} \int_{-1}^0 C_{11} u_{1x} d\eta = \frac{h^2}{2D_1} F_1(\sigma, p_1, p_2, m, n)$$

$$D_2^* = \frac{h^2}{2D_2} \int_0^1 C_{22} u_{2x} d\eta = \frac{h^2}{2D_2} F_2(\sigma, p_1, p_2, m, n)$$

Values of  $F_i$  are computed for different values of dimensionless parameters such as porous parameter  $\sigma$ , viscosity ratio  $m$ , and pressure gradients  $p_1, p_2$  and are shown in Table 2. Volumetric flow rate is also computed for variations of porous parameter, viscosity ratio, pressure gradients, and height of the channel, which are shown in Fig. 3. The numerical values of concentration for different porous parameters are shown in Table 3. Equations (33) and (34) are the well-known heat conduction equations, which can be solved easily for given initial conditions (Harleman et al., 1963).

### 2.2 Case 1b: Diffusion of a Tracer with Lower Wall Impermeable (Insulating) and Upper Wall Permeable (Conducting) Wall Conditions

To find the exact solutions of Eqs. (21) and (22), we require two more interface conditions along with boundary conditions (24), which are defined as

$$C_1 = C_2 \text{ and } \frac{\partial C_1}{\partial \eta} = D \frac{\partial C_2}{\partial \eta} \text{ at } \eta = 0 \quad (35)$$

**TABLE 2:** Values of effective dispersion coefficient for variations of porous parameter, viscosity ratio, and pressure gradients for impermeable wall conditions

$\sigma$	$F_1(\sigma, m, p)$	$F_2(\sigma, m, p)$	$F(\sigma, m, p)$
0.1	-0.7226690	0.0010511	-0.7216180
2	8.84313E-4	1.92822E-4	0.0010771
4	7.35777E-4	2.26777E-4	9.62554E-4
6	5.87131E-4	3.18392E-4	9.05523E-4
8	5.1204E-4	3.60536E-4	8.72575E-4
10	4.67126E-4	3.79267E-4	8.46393E-4
<i>m</i>			
0.1	0.0285818	0.0121764	0.0407582
1	8.84313E-4	1.92822E-4	0.0010771
2	-3.48332E-4	7.71969E-4	4.23637E-4
3	-0.0068575	0.0029188	-0.0039387
4	-0.0209342	0.0071638	-0.0137704
5	-0.0438161	0.0138319	-0.0299843
<i>p</i>			
-15	0.1989700	0.0433849	0.2423550
-10	0.0884313	0.0192822	0.1077130
-5	0.0221078	0.0048205	0.0269284
0.1	8.84313E-6	1.92822E-6	1.07713E-5
5	0.0221078	0.0048205	0.0269284
10	0.0884313	0.0192822	0.1077130
15	0.1989700	0.0433849	0.2423550

**TABLE 3:** Values of concentration for impermeable wall conditions

$\eta$	$D_1 = 0.148$ (Air – CO <sub>2</sub> ), $D_2 = 0.710$ (Air – H <sub>2</sub> )			$D_1 = 0.293$ (N <sub>2</sub> –H <sub>2</sub> O), $D_2 = 0.908$ (He – H <sub>2</sub> O)		
	$\sigma = 0.54$ (soil)	$\sigma = 0.65$ (silica grains)	$\sigma = 0.76$ (wire cramps)	$\sigma = 0.54$ (soil)	$\sigma = 0.65$ (silica grains)	$\sigma = 0.76$ (wire cramps)
-1.0	34.128598	15.109266	7.373876	17.239019	7.631984	3.724688
-0.8	34.145893	15.126174	7.390354	17.247755	7.640525	3.733012
-0.6	34.183522	15.163075	7.426443	17.266762	7.659164	3.751241
-0.4	34.223928	15.202965	7.465756	17.287172	7.679313	3.771099
-0.2	34.254500	15.233601	7.496460	17.302614	7.694788	3.786608
0.0	34.267426	15.247293	7.511001	17.309144	7.701704	3.793953
0.0	0	0	0	0	0	0
0.2	-0.001625	-0.001350	-0.001045	-0.001270	-0.001056	-8.17E-4
0.4	-0.007159	-0.006557	-0.005888	-0.005598	-0.005127	-0.004604
0.6	-0.015072	-0.014156	-0.013137	-0.011785	-0.011069	-0.010272
0.8	-0.022707	-0.021555	-0.020273	-0.017756	-0.016854	-0.015852
1.0	-0.026281	-0.025035	-0.023649	-0.020550	-0.019576	-0.018492

The solutions of Eqs. (21) and (22) satisfying conditions given in Eqs. (24) and (35) are as follows:

Region 1:

$$C_1 = \frac{h^2}{D_1 L} \frac{\partial C_1}{\partial \xi_1} \left( \frac{a_1 \cosh(\sigma\eta)}{\sigma^2} + \frac{a_2 \sinh(\sigma\eta)}{\sigma^2} + \frac{l_1 \eta^2}{2} \right) + b_1 \eta + b_2 \quad (36)$$

Region 2:

$$C_2 = \frac{h^2}{D_2 L} \frac{\partial C_2}{\partial \xi_2} \left( \frac{p_2 \eta^4}{24} + \frac{a_3 \eta^3}{6} + \frac{l_2 \eta^2}{2} \right) + b_3 \eta + b_4 \quad (37)$$

where  $b_1, b_2, b_3,$  and  $b_4$  are integrating constants. The expressions for  $C_1$  and  $C_2$  can also be written as

$$C_1 = \frac{h^2}{D_1 L} \frac{\partial C_1}{\partial \xi_1} C_{11} + \frac{h^2}{D_2 L} \frac{\partial C_2}{\partial \xi_2} C_{12} + C_{1c}$$

$$C_2 = \frac{h^2}{D_1 L} \frac{\partial C_1}{\partial \xi_1} C_{21} + \frac{h^2}{D_2 L} \frac{\partial C_2}{\partial \xi_2} C_{22} + C_{2c}$$

where

$$C_{11} = \left( \frac{a_1 \cosh(\sigma\eta)}{\sigma^2} + \frac{a_2 \sinh(\sigma\eta)}{\sigma^2} + \frac{l_1 \eta^2}{2} \right)$$

$$+ b_{11} \eta + b_{21};$$

$$C_{12} = b_{22}; \quad C_{1c} = b_{2c};$$

$$C_{21} = b_{31} \eta + b_{41};$$

$$C_{22} = \left( \frac{p_2 \eta^4}{24} + \frac{a_3 \eta^3}{6} + \frac{l_2 \eta^2}{2} \right) + b_{42}; \quad C_{2c} = b_{4c}$$

The volumetric flow rates at which the solute is transported across a section of the channel of unit breadth  $Q_1$  (Region 1) and  $Q_2$  (Region 2) using Eqs. (16), (17) and (36), (37) are

Region 1:

$$Q_1 = h \int_{-1}^0 C_1 u_{1x} d\eta = -(Q_{11} + Q_{12} + Q_{1c}) \quad (38)$$

where

$$Q_{11} = -\frac{h^3}{D_1 L} \frac{\partial C_1}{\partial \xi_1} \int_{-1}^0 C_{11} u_{1x} d\eta,$$

$$Q_{12} = -\frac{h^3}{D_2 L} \frac{\partial C_2}{\partial \xi_2} \int_{-1}^0 C_{12} u_{1x} d\eta$$

and

$$Q_{1c} = -h \int_{-1}^0 C_{1c} u_{1x} d\eta$$

Region 2:

$$Q_2 = h \int_0^1 C_2 u_{2x} d\eta = -(Q_{21} + Q_{22} + Q_{2c}) \quad (39)$$



where

$$Q_{21} = -\frac{h^3}{D_1 L} \frac{\partial C_1}{\partial \xi_1} \int_0^1 C_{21} u_{2x} d\eta,$$

$$Q_{22} = -\frac{h^3}{D_2 L} \frac{\partial C_2}{\partial \xi_2} \int_0^1 C_{22} u_{2x} d\eta$$

and

$$Q_{2c} = -h \int_0^1 C_{2c} u_{2x} d\eta$$

Following Taylor (1953), we assume that the variations of  $C_1$  and  $C_2$  with  $\eta$  are small compared with those in the longitudinal direction, and if  $C_{m1}$  and  $C_{m2}$  are the mean concentration over a section,  $\partial C_1 / \partial \xi_1$  and  $\partial C_2 / \partial \xi_2$  are indistinguishable from  $\partial C_{m1} / \partial \xi_1$  and  $\partial C_{m2} / \partial \xi_2$ , respectively, so that Eqs. (38) and (39) may be written as

Region 1:

$$Q_{11} = -D_{11}^* \frac{\partial C_{m1}}{\partial \xi_1}; \quad Q_{12} = -D_{12}^* \frac{\partial C_{m2}}{\partial \xi_2};$$

$$Q_{1c} = -D_{1c}^*$$

Region 2:

$$Q_{21} = -D_{21}^* \frac{\partial C_{m1}}{\partial \xi_1}; \quad Q_{22} = -D_{22}^* \frac{\partial C_{m2}}{\partial \xi_2}; \quad Q_{2c} = -D_{2c}^*$$

The fact that no material is lost in the process is expressed by the continuity equation for  $C_{m1}$  and  $C_{m2}$ , namely,

Region 1:

$$\frac{\partial Q_{11}}{\partial \xi_1} = -2 \frac{\partial C_{m1}}{\partial t}; \quad \frac{\partial Q_{12}}{\partial \xi_2} = -2 \frac{\partial C_{m2}}{\partial t} \quad (40)$$

Region 2:

$$\frac{\partial Q_{21}}{\partial \xi_1} = -2 \frac{\partial C_{m1}}{\partial t}; \quad \frac{\partial Q_{22}}{\partial \xi_2} = -2 \frac{\partial C_{m2}}{\partial t} \quad (41)$$

Equations (40) and (41) using Eqs. (38) and (39) become

Region 1:

$$\frac{\partial C_{m1}}{\partial t} = \frac{D_{11}^*}{2} \frac{\partial^2 C_{m1}}{\partial \xi_1^2}; \quad \frac{\partial C_{m2}}{\partial t} = \frac{D_{12}^*}{2} \frac{\partial^2 C_{m2}}{\partial \xi_2^2} \quad (42)$$

Region 2:

$$\frac{\partial C_{m1}}{\partial t} = \frac{D_{21}^*}{2} \frac{\partial^2 C_{m1}}{\partial \xi_1^2}; \quad \frac{\partial C_{m2}}{\partial t} = \frac{D_{22}^*}{2} \frac{\partial^2 C_{m2}}{\partial \xi_2^2} \quad (43)$$

which are the equations governing the longitudinal dispersion, where

$$D_{11}^* = \frac{h^2}{2D_1} \int_{-1}^0 C_{11} u_{1x} d\eta = \frac{h^2}{2D_1} F_{11}(\sigma, p_1, p_2, m, n)$$

$$D_{12}^* = \frac{h^2}{2D_2} \int_{-1}^0 C_{12} u_{1x} d\eta = \frac{h^2}{2D_2} F_{12}(\sigma, p_1, p_2, m, n)$$

$$D_{1c}^* = h \int_{-1}^0 C_{1c} u_{1x} d\eta = h F_{1c}$$

$$D_{21}^* = \frac{h^2}{2D_1} \int_0^1 C_{21} u_{2x} d\eta = \frac{h^2}{2D_1} F_{21}(\sigma, p_1, p_2, m, n);$$

$$D_{22}^* = \frac{h^2}{2D_2} \int_0^1 C_{22} u_{2x} d\eta = \frac{h^2}{2D_2} F_{22}(\sigma, p_1, p_2, m, n);$$

$$D_{2c}^* = h \int_0^1 C_{2c} u_{2x} d\eta = h F_{2c}$$

Values of  $F_{ii}$  are computed for different values of dimensionless parameters such as porous parameter  $\sigma$ , viscosity ratio  $m$ , and pressure gradients  $p_1$ ,  $p_2$  and are tabulated in Table 4. Volumetric flow rate is also computed for variations of porous parameter, viscosity ratio, pressure gradients, and height of the channel, which are shown in Fig. 4. Equations (42) and (43) are the well-known heat conduction equations, which can be solved easily.

### 2.3 Case 2a: Diffusion of a Tracer for the Channel Filled with Porous Matrix (One-Fluid Model) with Impermeable Wall Conditions

To validate the results obtained for composite porous medium (two-fluid model), the problem is solved considering the channel filled with only porous medium (one-fluid model), which was studied by Rudraiah and Ng (2007).

The concentration equation for a one-fluid model using Taylor (1953) becomes

$$\frac{\partial^2 C}{\partial \eta^2} = \frac{h^2}{DL} \frac{\partial C}{\partial \xi} u_x \quad (44)$$

where

$$u_x = -\frac{p}{\sigma^2} \left( \frac{\tanh(\sigma)}{\sigma} - \frac{\cosh(\sigma\eta)}{\cosh(\sigma)} \right)$$

**TABLE 4:** Values of effective dispersion coefficient for variations of porous parameter, viscosity ratio, and pressure gradients for insulating and conducting wall conditions

$\sigma$	$F_1(\sigma, m, p)$	$F_2(\sigma, m, p)$	$F(\sigma, m, p)$
0.1	9.80459E-4	0.0011268	0.0021073
2	-0.0138537	0.0147989	9.45194E-4
4	-0.0214911	0.0223342	8.43056E-4
6	-0.0232052	0.0240655	8.6037E-4
8	-0.0234677	0.0243197	8.52049E-4
10	-0.0233521	0.0241876	8.35574E-4
<i>m</i>			
0.1	-0.1651240	0.2097070	0.0445838
1	-0.0138537	0.0147989	9.45194E-4
2	0.0144200	-0.0128954	0.0015246
3	0.0500962	-0.0459284	0.0041679
4	0.0937380	-0.0842678	0.0094702
5	0.1444750	-0.1266000	0.0178747
<i>p</i>			
-15	0.3557220	-0.1430530	0.2126690
-10	0.2063320	-0.1118130	0.0945194
-5	0.0877582	-0.0641283	0.0236299
0.1	-0.0014408	0.0014503	9.45194E-6
5	-0.0569424	0.0805723	0.0236299
10	-0.0830690	0.1775880	0.0945194
15	-0.0783799	0.2910490	0.2126690

The solution of Eq. (44) using boundary conditions  $\partial C/\partial \eta = 0$  at  $\eta = \pm 1$  is

$$C = \frac{h^2}{DL} \frac{\partial C}{\partial \xi} \frac{p}{\sigma^2} \left( \frac{\cosh(\sigma \eta)}{\sigma^2 \cosh(\sigma)} - \frac{\eta^2 \tanh(\sigma)}{2\sigma} \right) + b_2 \quad (45)$$

where the constant  $b_2$  is to be determined using the entry condition.

The volumetric flow rate at which the solute is transported across a section of the channel of unit breadth is given by

$$Q = -2D^* \frac{\partial C_m}{\partial \xi} \quad (46)$$

where

$$D^* = \frac{h^2 p^2}{D} \left[ \tanh^2(\sigma) \left( -\frac{2}{3\sigma^6} - \frac{4}{\sigma^8} \right) + \frac{2 \tanh(\sigma)}{\sigma^7} + \frac{1}{\sigma^6 \cosh^2(\sigma)} \left( 1 + \frac{\sinh(2\sigma)}{2\sigma} \right) \right] = \frac{h^2}{D} F(\sigma, p) \quad (47)$$

and  $F(\sigma, p)$  represents the terms in square brackets of Eq. (47) multiplied by  $p^2$  of Eq. (47).

The values for velocity are computed and are shown in Table 1 for a composite porous medium and for a one-fluid model considering  $p_1 = p_2 = 1, m = 1, n = 1$ . It is seen that for  $\sigma = 0.01$ , the values of velocity agree very well, whereas for large values of  $\sigma$ , the values of velocity do not agree for two-fluid and one-fluid models. The values of dispersion coefficient  $F(\sigma, p)$  are evaluated and shown in Table 6.

**2.4 Case 2b: Diffusion of a Tracer for the Channel Filled with Porous Matrix (One-Fluid Model) with Lower Wall Impermeable and Upper Wall Permeable Wall Conditions**

The solution of Eq. (44) using boundary conditions

$$\frac{\partial C}{\partial \eta} = 0 \text{ at } \eta = -1 \text{ and } C = 1 \text{ at } \eta = 1$$

is

$$C = \frac{h^2}{DL} \frac{\partial C}{\partial \xi} \frac{p}{\sigma^2} \left( \frac{\cosh(\sigma \eta)}{\sigma^2 \cosh(\sigma)} - \frac{\eta^2 \tanh(\sigma)}{2\sigma} \right) + b_1 \eta + b_2 \quad (48)$$

Following the analysis as explained in case 2a, the values of the dispersion coefficient  $F(\sigma, p)$  are evaluated and shown in Table 6.

**2.5 Case 3a: Diffusion of a Tracer in the Absence of Porous Matrix with Impermeable Wall Conditions (Two-Fluid Model)**

To validate the results of the present model, the problem is solved in the absence of porous medium and compared with the results of Gupta and Gupta (1972). The solutions of Eqs. (5) and (6) using boundary and interface conditions (7) in the absence of porous parameter  $\sigma$  become

$$u_1 = \frac{p_1 \eta^2}{2} + a_1 \eta + a_2 \quad (49)$$

$$u_2 = \frac{p_2 \eta^2}{2} + a_3 \eta + a_4 \quad (50)$$

The average velocities as defined in Eqs. (10) and (11) in the absence of porous parameter become

$$\bar{u}_1 = \frac{1}{2} \left( \frac{p_1}{6} - \frac{a_1}{2} + a_2 \right); \quad \bar{u}_2 = \frac{1}{2} \left( \frac{p_1}{6} + \frac{a_3}{2} + a_4 \right)$$

The solutions of Eqs. (21) and (22) in the absence of porous parameter  $\sigma$  yield

$$C_1 = \frac{h^2}{D_1 L} \frac{\partial C_1}{\partial \xi_1} \left( \frac{p_1 \eta^4}{24} + \frac{a_1 \eta^3}{6} + \frac{lc_1 \eta^2}{2} + b_1 \eta \right) + C_{01} \quad (51)$$

$$C_2 = \frac{h^2}{D_2 L} \frac{\partial C_2}{\partial \xi_2} \left( \frac{p_2 \eta^4}{24} + \frac{a_3 \eta^3}{6} + \frac{lc_2 \eta^2}{2} + b_3 \eta \right) + C_{02} \quad (52)$$

where  $C_{01}$  and  $C_{02}$  are constants to be determined using entry conditions.

The volumetric flow rates at which the solute is transported across a section of the channel of unit breadth  $Q_1$  (Region 1) and  $Q_2$  (Region 2) and the evaluation of effective dispersion coefficients  $F_i$  are evaluated as explained in case 1a in the absence of porous parameter. The values of  $F_i$  ( $p_1, p_2, m, n$ ) are computed for different values of the dimensionless parameters  $p_i$  and  $m$  and are shown in Table 7.

### 2.6 Case 3b: Diffusion of a Tracer in the Absence of Porous Matrix with Impermeable Wall Conditions (One-Fluid Model)

The solution for concentration obtained by Gupta and Gupta (1972) in the absence of chemical reaction is

$$C = \frac{h^2}{DL} \frac{\partial C}{\partial \xi} \left( \frac{p}{24} \eta^4 - \frac{p}{12} \eta^2 \right) + C_0 \quad (53)$$

where  $C_0$  is a constants to be determined using entry conditions.

The volumetric flow rate in which the solute is transported across a section of the channel of unit breadth is

$$Q = h \int_{-1}^1 C u_x d\eta = -\frac{h^2 p^2}{D} \frac{\partial C}{\partial \xi} \left( \frac{2}{945} \right) = D^* \frac{\partial C}{\partial \xi} \quad (54)$$

where

$$D^* = \frac{h^2 p^2}{D} \frac{2}{945}$$

By comparing with Fick's law of diffusion,  $D^*$  agrees with the results of Wooding (1960), where  $p$  is a nondimensional pressure gradient.

All the constants appearing previously are defined in the Appendix.

## 3. RESULTS AND DISCUSSION

This problem is concerned with the longitudinal dispersion of a solute subject to molecular diffusion when it is introduced into a channel for a composite porous medium following a Taylor diffusion model.

To find the average velocity in both regions of the channel, no-slip conditions at the boundaries and continuity of velocity and shear stress are assumed at the interface. The closed-form solutions are obtained for the concentration in Region 1 and Region 2. The volumetric flow

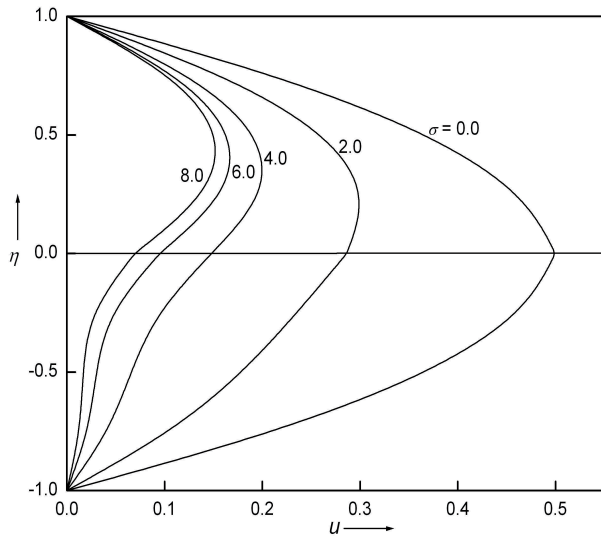
rates in both the regions of the channel are also found. The effective dispersion coefficient in each region is also evaluated, and the values are tabulated for variations of governing parameters such as porous parameter, viscosity ratio, and pressure gradients. The distribution of solute concentration is analyzed for two types of boundary conditions: the first one is connected with an insulated type of boundary condition and the other is lower wall insulating and upper wall conducting. The values for viscosity ratio, pressure gradient, and porous parameter are fixed as 1, 1, 2, respectively, except for varying parameters in all the tables and graphs.

### 3.1 Case 1a: Diffusion of a Tracer with Impermeable Wall Conditions

The effect of porous parameter  $\sigma$  on the velocity for composite porous medium (two-fluid model) is evaluated and shown in Table 1. The problem of dispersion in a porous medium with and without chemical reaction was analyzed by Rudraiah and Ng (2007). In this problem the channel is filled with only a porous matrix. The velocity distribution is computed and shown in Table 1 (one-fluid model). It is seen that for small values of porous parameter, the velocity distribution is the same in the porous region for the two-fluid model (present model) and one-fluid model (Rudraiah and Ng, 2007). As the porous parameter  $\sigma$  increases, the values of velocity are close near the lower wall but vary near the interface and in the viscous fluid region.

Figure 2 displays the effect of porous parameter  $\sigma$  on the velocity field for composite porous medium. It is observed that the velocity decreases as the porous parameter  $\sigma$  increases. For large porous parameter, the frictional drag resistance against the flow in the porous region is very large, and as a result, the velocity is very small in the porous region. The velocity in a clear fluid region also decreases with an increase in the value of the porous parameter. This is due to the coupling effect.

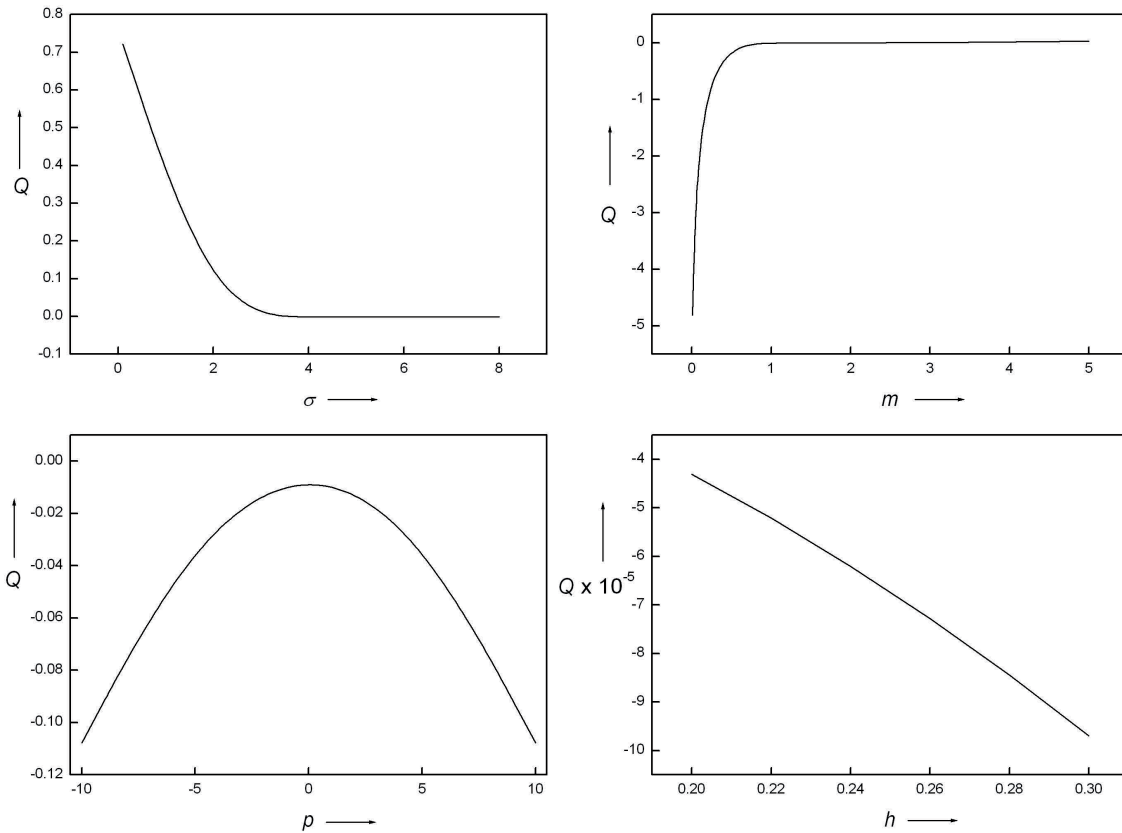
The effects of porous parameter  $\sigma$ , viscosity ratio  $m$ , pressure gradient  $p$  ( $= p_1 = p_2$ ), and the height of the channel  $h$  on the volumetric flow rate is shown in Fig. 3. It is seen that as the porous parameter  $\sigma$  increases, volumetric flow rate decreases for small values of  $\sigma$  and remains almost invariant for  $\sigma > 3$ . As the viscosity ratio  $m$  increases, volumetric flow rate increases in magnitude for the values of  $m$  from 0 to 0.5 (approximately) and then remains constant for large values of viscosity ratio  $m$ . Volumetric flow rate is symmetric for negative and positive values of pressure gradient  $p$ , and the optimum flow rate



**FIG. 2:** Velocity profiles for different values of porous parameter  $\sigma$

is attained in the absence of a pressure gradient in magnitude. As the height of the channel  $h$  increases, the volumetric flow rate decreases in magnitude for all values of  $h$ .

The effective dispersion coefficients  $F_1(\sigma, m, p)$ ,  $F_2(\sigma, m, p)$ , and  $F(F_1 + F_2)$  for variations of porous parameter  $\sigma$ , viscosity ratio  $m$ , and pressure gradient  $p$  are shown in Table 2. As the porous parameter  $\sigma$  increases, the effective dispersion coefficient  $F$  decreases very rapidly, which is similar to the result obtained by Rudraiah and Ng (2007). This is due to the fact that as  $\sigma$  grows, the velocity profiles continue to flatten out and tend to plug flow as  $\sigma \rightarrow \infty$ . This flattening is the cause for the decrease in  $F$  with an increase in  $\sigma$ . As the viscosity ratio  $m$  increases, the effective dispersion coefficient  $F$  decreases for  $m \leq 1$  and increases in magnitude for  $m \geq 1$ . The effective dispersion coefficient  $F$  is symmetric for pressure gradient  $p > 0$  and for  $p < 0$ . Further effective dispersion coefficient  $F$  increases as  $p$  increases for values of  $p > 0$ ; this is due to the fact that from the



**FIG. 3:** Volumetric flow rate  $Q$  for impermeable wall conditions versus porous parameter  $\sigma$ , viscosity ratio  $m$ , pressure gradient  $p$ , and height of the channel  $h$

equation of motion, it is clear that as  $p$  increases, velocity increases, which causes the increase in  $F$ .

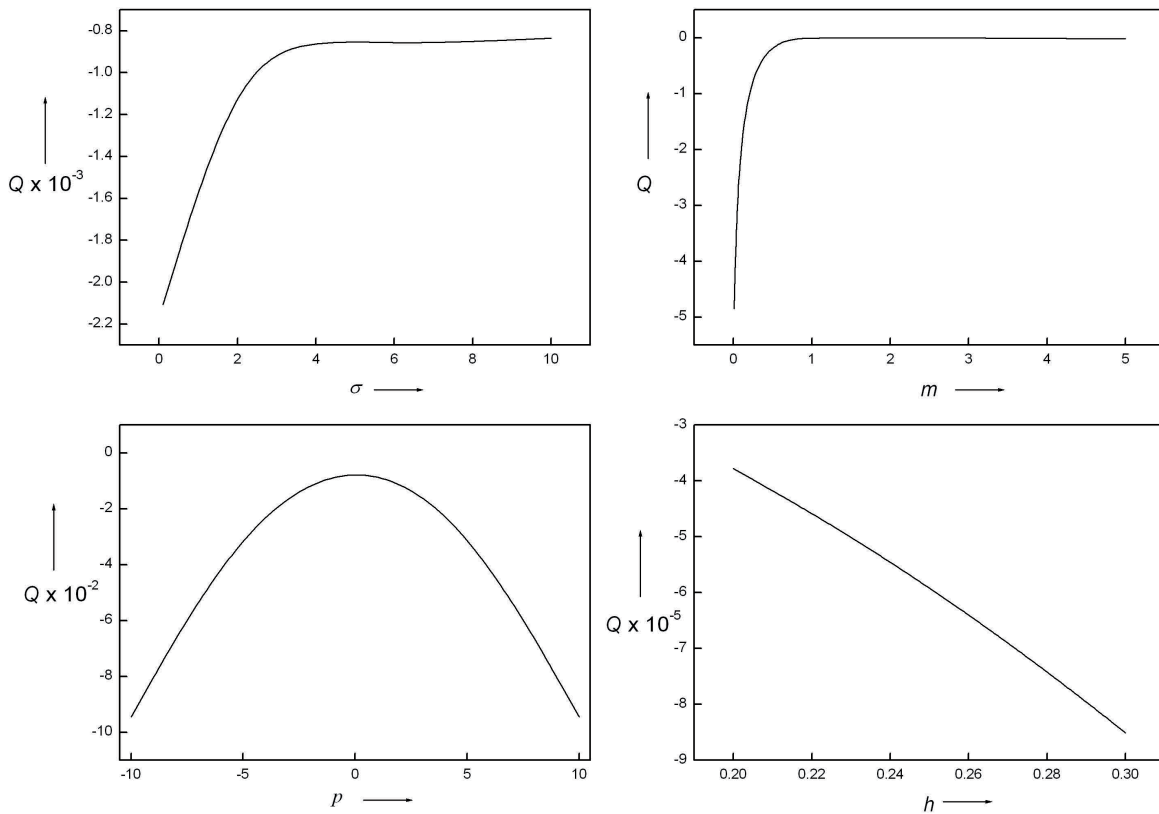
To understand the nature of the distribution of concentration for the real field, physical numbers for the porous parameter and diffusivity coefficients are chosen and shown in Table 3. The experimental values of diffusion coefficients in gases at one atmosphere are chosen from Cussler (1998). It is interesting to note that for any combination of gases, as the porous parameter  $\sigma$  increases, the concentration decreases in both of the regions. This is due to the fact that an increase in the grain size increases the permeability  $\kappa$  and hence decreases  $\sigma$ , which in turn reduces the value of the diffusion coefficient (this behavior is in conformity with the experimental results of Harleman et al., 1963), which results in reduction of the concentration. Two different fluids are chosen in each region such as carbon dioxide and hydrogen, with both the regions filled with air. Another combination of nitrogen and helium is chosen with both regions filled with water. It is seen that the concentration is high for

the channel filled with air when compared to the channel filled with water in both of the regions for variations in the diffusivity coefficient.

### 3.2 Case 1b: Diffusion of a Tracer with Lower Wall Impermeable (Insulating) and Upper Wall Permeable (Conducting) Wall Conditions

The effects of porous parameter  $\sigma$ , viscosity ratio  $m$ , pressure gradient  $p$  ( $= p_1 = p_2$ ), and the height of the channel  $h$  on the volumetric flow rate are shown in Fig. 4. It is seen that the volumetric flow rate increases in magnitude for small values of  $\sigma$  and remains constant for values of  $\sigma > 4$ . The effect of viscosity ratio  $m$ , pressure gradient  $p$ , and height of the channel  $h$  shows a similar effect on the volumetric flow rate as in the case of insulating wall conditions (case 1a).

The effects of porous parameter  $\sigma$ , viscosity ratio  $m$ , and pressure gradient  $p$  on the effective dispersion coef-



**FIG. 4:** Volumetric flow rate  $Q$  for lower wall impermeable and upper wall permeable wall conditions versus porous parameter  $\sigma$ , viscosity ratio  $m$ , pressure gradient  $p$ , and height of the channel  $h$

ficient are shown in Table 4. As the porous parameter  $\sigma$  increases, the effective dispersion coefficient  $F$  decreases, but the magnitude of suppression is very small when compared to insulating wall boundary conditions. The effect of viscosity ratio  $m$  is to decrease  $F$  for  $m \leq 1$  and increase  $F$  for  $m \geq 1$ . The effect of pressure gradient  $p$  shows a similar nature on the effective dispersion coefficient  $F$  as insulating wall conditions (case 1a).

The effect of the porous parameter on the concentration of soil, silica grains, and wire cramps using the values of the diffusivity coefficient from experimental values is shown in Table 5. The effect of  $\sigma$  on the concentration is not effective when compared to impermeable wall conditions. However, the values of the concentration are less for air when compared to water, which is in contradiction to impermeable wall conditions.

**3.3 Cases 2a and 2b: Diffusion of a Tracer for the Channel Filled with Porous Matrix (One-Fluid Model) with Impermeable Wall Conditions and Lower Wall Impermeable and Upper Wall Permeable Wall Boundary Conditions**

The problem of dispersion in porous media with and without chemical reaction was analyzed by Rudraiah and Ng (2007). In this problem, the channel is filled with only porous matrix. The effective dispersion coefficients for this model are evaluated for impermeable wall conditions

and with lower wall impermeable and upper wall permeable wall conditions and are shown in Table 6.

It is observed from Table 1 that as the porous parameter  $\sigma$  increases, the velocity field differs for composite porous media (two-fluid model) with the results of Rudraiah and Ng (2007) (one-fluid model). Hence the results obtained for effective dispersion coefficient cannot be tallied with the results of Rudraiah and Ng (2007). To validate the present model, the problem is solved by considering the channel filled with saturated porous media in both the regions with different permeabilities  $\kappa_1$  and  $\kappa_2$ . The effective dispersion coefficient is evaluated for various values of porous parameter  $\sigma$  ( $= \sigma_1 = \sigma_2$ ) for the two-fluid model and is shown in Table 6. The effective dispersion coefficient  $F$  for the one-fluid model is computed using Rudraiah and Ng (2007). It is observed that the effective dispersion coefficient  $F$  ( $= F_1 = F_2$ ) agrees very well for all values of porous parameter  $\sigma$  of the present model (two-fluid model) with the result of Rudraiah and Ng (2007) (one-fluid model).

**3.4 Cases 3a and 3b: Diffusion of a Tracer in the Absence of Porous Matrix with Impermeable Wall Conditions for Two-Fluid Model and One-Fluid Model**

To justify the present model, we solved the problem in the absence of porous matrix for the channel filled with two different viscous fluids and compared the results for the

**TABLE 5:** Values of concentration with lower wall impermeable and upper wall permeable wall conditions

$\eta$	$D_1 = 0.148$ (Air - CO <sub>2</sub> ), $D_2 = 0.710$ (Air - H <sub>2</sub> )			$D_1 = 0.293$ (N <sub>2</sub> -H <sub>2</sub> O), $D_2 = 0.908$ (He - H <sub>2</sub> O)		
	$\sigma = 0.54$ (soil)	$\sigma = 0.65$ (silica grains)	$\sigma = 0.76$ (wire cramps)	$\sigma = 0.54$ (soil)	$\sigma = 0.65$ (silica grains)	$\sigma = 0.76$ (wire cramps)
-1.0	0.887452	0.887008	0.886524	0.950425	0.949856	0.949228
-0.8	0.904747	0.903917	0.903002	0.959161	0.958397	0.957551
-0.6	0.942376	0.940817	0.939091	0.978168	0.977036	0.975780
-0.4	0.982782	0.980707	0.978404	0.998578	0.997185	0.995638
-0.2	1.013355	1.011343	1.009108	1.014021	1.012660	1.011147
0.0	1.026281	1.025035	1.023649	1.020550	1.019576	1.018492
0.0	1.026281	1.025035	1.023649	1.020550	1.019576	1.018492
0.2	1.024656	1.023685	1.022604	1.019280	1.018520	1.017675
0.4	1.019122	1.018478	1.017761	1.014952	1.014449	1.013888
0.6	1.011209	1.010879	1.010512	1.008765	1.008507	1.008220
0.8	1.003574	1.003480	1.003377	1.002794	1.002721	1.002640
1.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

**TABLE 6:** Values of effective dispersion coefficient for various values of porous parameter  $\sigma$  for two-fluid and one-fluid models

$\sigma$	Two-fluid model (present model)			One-fluid model (Rudraiah and Ng, 2007)
	$F_1(\sigma, m, p)$	$F_2(\sigma, m, p)$	$F(\sigma, m, p)$	$F(\sigma, m, p)$
Impermeable wall boundary conditions				
0.1	0.0020982	0.0020982	0.0041963	0.0041963
1	0.0010148	0.0010148	0.0020296	0.0020296
2	2.51743E-4	2.51743E-4	5.03486E-4	5.03486E-4
3	6.53142E-5	6.53142E-5	1.30628E-4	1.30628E-4
4	2.00927E-5	2.00927E-5	4.01854E-5	4.01854E-5
5	7.24446E-6	7.24446E-6	1.44889E-5	1.44889E-5
Lower wall impermeable and upper wall permeable boundary conditions				
0.1	0.0010491	0.0010491	0.0020982	0.0020982
2	1.25871E-4	1.25871E-4	2.51743E-4	2.51743E-4
4	1.00464E-5	1.00464E-5	2.00927E-5	2.00927E-5
6	1.48823E-6	1.48823E-6	2.97647E-6	2.97647E-6
8	3.37759E-7	3.37759E-7	6.75518E-7	6.75518E-7
10	1.01667E-7	1.01667E-7	2.03333E-7	2.03333E-7

channel filled with one viscous fluid (Gupta and Gupta, 1972) in the absence of chemical reactions. The effects of viscosity ratio  $m$  and pressure gradient  $p$  on the effective dispersion coefficient  $F$  is computed for a two-fluid model and is shown in Table 7. As  $m$  increases,  $F$  de-

creases for the values of  $m \leq 1$  and increases as  $m$  increases for the values of  $m \geq 1$ . The effective dispersion coefficient  $F$  is symmetric for  $p > 0$  and for  $p < 0$ . Furthermore, it is observed that the effective dispersion coefficient  $F$  agrees with two-fluid (present model) and one-

**TABLE 7:** Values of effective dispersion coefficient for various values of viscosity ratio  $m$  and pressure gradient  $p$

$m$	$F_1(m, p)$	$F_2(m, p)$	$F(m, p)$	
0.1	0.1970520	0.0695106	0.2665620	
1	0.0010582	0.0010582	0.0021164	
2	0.0012463	0.0042121	0.0054584	
3	0.0031994	0.0113013	0.0145007	
4	0.0068564	0.0229046	0.0297609	
5	0.0123545	0.0393175	0.0516720	
6	0.0197790	0.0606913	0.0804702	
$p$				
-15	0.2380950	0.2380950	0.4761900	
-10	0.1058200	0.1058200	0.2116400	
-5	0.0264550	0.0264550	0.0529101	
0.1	1.0582E-5	1.0582E-5	2.1164E-5	
5	0.0264550	0.0264550	0.0529101	
10	0.1058200	0.1058200	0.2116400	
15	0.2380950	0.2380950	0.4761900	
Two-fluid model (present model)				One-fluid model (Gupta and Gupta, 1972)
$m = p$	$F_1(m, p)$	$F_2(m, p)$	$F(m, p)$	$F(m, p)$
1	0.0010582	0.0010582	0.0021164	0.0021164

fluid models (Gupta and Gupta, 1972) for equal values of viscosity ratio and pressure gradients.

#### 4. CONCLUSIONS

The problem of solute dispersion in laminar flow of a composite porous medium between two parallel plates was studied using Taylor's dispersion model. Exact solutions were obtained for two different types of boundary conditions such as insulating and insulating permeable wall boundary conditions.

The results obtained for composite porous media were as follows:

1. The effective dispersion coefficient decreases as the porous parameter increases for both insulating and insulating permeable wall boundary conditions.
2. The effective dispersion coefficient decreases as the viscosity ratio increases for values of viscosity ratio less than 1 and increases in magnitude for values of viscosity ratio greater than 1 for both insulating and insulating permeable wall boundary conditions.
3. The effective dispersion coefficient was symmetric for the values of pressure gradient  $p < 0$ ,  $p > 0$  and the effective dispersion coefficient increases as pressure gradient increases for  $p > 0$  for both insulating and insulating permeable wall boundary conditions.
4. The effect of porous parameter for soil, silica grain, and wire cramps was to decrease the concentration, which was a similar result to that obtained experimentally by Harleman et al. (1963). Furthermore, it was also observed that the concentration distribution was high for air when compared to water for insulating wall boundary conditions. For insulating permeable wall boundary conditions, the effect of porous parameter on the concentration for the soil, silica grains, and wire cramps was not effective on the concentration distribution when compared to insulating wall boundary conditions. The distribution of concentration for air was less when compared to water.

To justify the results of the present model (two-fluid model), the problem was solved considering the channel filled with different porous matrices in both the regions and the effective dispersion coefficient was computed. The effective dispersion coefficient was computed from Rudraiah and Ng (2007) (one-fluid model). There was a good agreement between the present model and the model

of Rudraiah and Ng (2007) in the presence of porous matrix.

To further justify the results, the problem was solved in the absence of porous matrix for a two-fluid model, and the results were in good agreement with the results of Gupta and Gupta (1972) in the absence of chemical reaction for a one-fluid model.

#### ACKNOWLEDGMENT

One of the authors, J.C.U., thanks UGC–New Delhi for financial support under UGC–Major Research Project.

#### REFERENCES

- Alzami, B. and Vafai, K., Analysis of fluid flow and heat transfer interfacial conditions between a porous medium and a fluid layer, *Int. J. Heat Mass Transfer*, vol. **44**, pp. 1735–1749, 2001.
- Bear, J. and Bechmat, Y., A unified approach transport phenomena in porous media, underground storage and mixing project, Progr. Rep. 3, Haifa Technion. Israel Inst. Technol. Hydraulic Laboratory, P.N. 1/65, 1965.
- Beavers, G. and Joseph, D. D., Boundary conditions at a naturally permeable wall, *J. Fluid Mech.*, vol. **30**, pp. 197–207, 1967.
- Brinkman, H. C., A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res.*, vol. **A1**, pp. 27–34, 1947.
- Catton, I., *Natural Convection Heat Transfer in Porous Media: Fundamentals and Applications*, ed. S. Kakac, W. Aung, and R. Viskanta, Hemisphere, Washington, DC, 1985.
- Cussler, E. L., *Diffusion Mass Transfer in Fluid Systems*, Cambridge University Press, New York, 1998.
- Derjani, G., Taim, M. E., and Narusawa, U., Effect of boundary conditions on thermal instability of superposed porous and fluid layer, in *Natural Convection in Enclosures*, ed. R. S. Figliola and I. Catton, pp. 83–89, ASME, New York, 1986.
- Gupta, P. and Gupta, A. S., Effect of homogeneous and heterogeneous reactions on the dispersion of solute in a laminar flow between two plates, *Proc. R. Soc. London, Ser. A*, vol. **330**, pp. 59–63, 1972.
- Harleman, D. R. F., Melhorn, P. F., and Rumer, R. R., Dispersion-permeability correlation in porous media, *J. Hydraulic Div. Am. Soc. Civil Eng.*, vol. **89**, pp. 67–85, 1963.
- Kaviany, M., *Principles of Heat Transfer in Porous Media*, Springer, New York, 1999.
- Kuznetsov, A. V., Analytical study of forced convection in partly porous configurations, in *Handbook of Porous Media*, ed. K. Vafai, pp. 269–312, Marcel Dekker, New York, 2000.



- Malashetty, M. S., Umavathi, J. C., and Prathap Kumar, J., Convective flow and heat transfer in a composite porous medium, *J. Porous Media*, vol. **4**, no. 1, pp. 15–22, 2001.
- Masuoka, T., Convective currents in a horizontal layer divided by a permeable wall, *Bull. Jpn. Soc. Mech. Eng.*, vol. **17**, pp. 225–237, 1974.
- Neale, G. and Nader, W., Practical significance of Brinkman's extension of Darcy's law: Couples parallel flows within a channel and a boundary porous medium, *Can. J. Chem. Eng.*, vol. **52**, pp. 475–478, 1974.
- Ng, C. O., Rudraiah, N., Nagaraj, C., and Nagaraj, H. N., Electrohydrodynamic dispersion of macro molecular components in nanostructured biological bearing, *J. Energy Heat Mass Transfer*, vol. **27**, pp. 39–64, 2005.
- Nield, D. A., Onset of convection in a fluid layer overlying a layer of a porous medium, *J. Fluid Mech.*, vol. **81**, pp. 513–522, 1977.
- Nield, D. A., The boundary correction for the Rayleigh–Darcy problem: Limitations of the Brinkman equation, *J. Fluid Mech.*, vol. **128**, pp. 37–46, 1983.
- Nield, D. A. and Bejan, A., *Convection in Porous Media*, Springer, New York, 1999.
- Posner, A. and Gill, W. N., Miscible displacement with combined free and forced convection-laminar flow in a vertical tube, *AIChE J.*, vol. **19**, pp. 151–158, 1973.
- Prathap Kumar, J., Umavathi, J. C., Pop, I., and Basavaraj M. Biradar, Fully developed mixed convection flow in a vertical channel containing porous and fluid layer with isothermal or isoflux boundaries, *Transport Porous Media*, vol. **80**, pp. 117–135, 2009.
- Rudraiah, N., Dispersion of soluble matter in laminar flow through porous media between two parallel plates, *Bangalore Univ. J.*, vol. **2**, pp. 1–10, 1976.
- Rudraiah, N. and Ng, C. O., Dispersion in porous media with and without reaction: A review, *J. Porous Media*, vol. **10**, no. 3, pp. 219–248, 2007.
- Rudraiah, N. and Veerabhadraiah, R., Temperature distribution in Couette flow past a permeable bed, *Proc. Ind. Acad. Sci.*, vol. **86**, pp. 537–547, 1977.
- Saffman, P. G., On the boundary condition at the interface of a porous medium, *Stud. Appl. Math.*, vol. **1**, no. 2, pp. 93–101, 1971.
- Simpson, E. S., *Flow through Porous Media*, ed. R. J. M. Dewiest, Academic, New York, 1969.
- Somerton, C. W. and Catton, I., On the thermal instability of superimposed porous and fluid layers, *ASME J. Heat Transfer*, vol. **104**, pp. 160–165, 1982.
- Sueiu, A. N., Iwatsbo, T., and Matsuda, M., Theoretical investigation of an artificial joint with micro-pocket-covered component and biphasic cartilage on the opposite articulating surface, *ASME J. Biomech. Eng.*, vol. **125**, pp. 425–433, 2003.
- Taber, J. J., Research on enhanced oil recovery—past, present and future, *Pure Appl. Chem.*, vol. **52**, pp. 1323–1347, 1980.
- Taylor, G. I., Dispersion of soluble matter in solvent flowing slowly through a tube, *Proc. R. Soc. London, Ser. A*, vol. **219**, pp. 186–203, 1953.
- Umavathi, J. C., Chamkha, A. J., and Sridhar, K. S. R., Generalised plain Couette flow and heat transfer in a composite channel, *Transport Porous Media*, vol. **85**, pp. 157–169, 2010.
- Vafai, K., *Handbook of Porous Media*, Marcel Dekker, New York, 2000.
- Vafai, K. and Kim, S. J., Fluid mechanics of the interface region between a porous medium and a fluid layer—an exact solution, *Int. J. Heat Fluid Flow*, vol. **11**, pp. 254–256, 1990.
- Wooding, R. A., Instability of a viscous liquid of variable density in a vertical Hele-Shaw cell, *J. Fluid Mech.*, vol. **7**, pp. 501–515, 1960.

## APPENDIX

### A1. Case 1a: Diffusion of a Tracer with Impermeable (Insulating) Wall Conditions

$$a_3 = \frac{-\sigma}{m n (\sigma \cosh(\sigma) + m \sinh(\sigma))}$$

$$\times \left\{ \frac{p_1}{\sigma^2} [1 - \cosh(\sigma)] + \frac{p_2}{2} m n \cosh(\sigma) \right\}$$

$$a_4 = -a_3 - \frac{p_2}{2}; \quad a_1 = m n a_4 + \frac{p_1}{\sigma^2}$$

$$a_2 = \frac{1}{\sinh(\sigma)} \left[ a_1 \cosh(\sigma) - \frac{p_1}{\sigma^2} \right]$$

$$lc_1 = -\frac{1}{2} \left( \frac{a_2}{\sigma} + \frac{a_1 \sinh(\sigma)}{\sigma} - \frac{a_2 \cosh(\sigma)}{\sigma} - \frac{p_1}{\sigma^2} + \frac{p_2}{6} + \frac{a_3}{2} + a_4 \right)$$

$$l_1 = \frac{-p_1}{\sigma^2} + lc_1; \quad l_2 = a_4 + lc_1; \quad Z_1 = \frac{h^2}{D_1 L} \frac{\partial C_1}{\partial \xi_1}$$

$$Z_2 = \frac{h^2}{D_2 L} \frac{\partial C_2}{\partial \xi_2}; \quad b_1 = Z_1 b_{11}; \quad b_3 = Z_2 b_{32}$$

$$b_{11} = \frac{a_1 \sinh(\sigma)}{\sigma} - \frac{a_2 \cosh(\sigma)}{\sigma} + l_1$$

$$\begin{aligned}
 b_2 &= 0; \quad b_{32} = \frac{p_2}{6} + \frac{a_3}{2} + l_2; \quad b_4 = 0; \\
 Q_1 &= Z_1 h \left\{ \frac{a_1 a_2}{2\sigma^3} + \frac{2l_1 a_2}{\sigma^3} - \frac{a_1 b_{11}}{\sigma^2} + \frac{\sinh(2\sigma)}{2\sigma} \right. \\
 &\times \left( \frac{a_1^2}{2\sigma^2} + \frac{a_2^2}{2\sigma^2} \right) - \frac{a_1 a_2 \cosh(2\sigma)}{2\sigma^3} + \sinh(\sigma) \\
 &\times \left[ \frac{l_1 a_1}{\sigma^3} + \frac{l_1 a_1}{2} \left( \frac{1}{\sigma} + \frac{2}{\sigma^3} \right) + \frac{l_1 a_2}{\sigma^2} - \frac{a_1 b_{11}}{\sigma} - \frac{a_2 b_{11}}{\sigma^2} \right] \\
 &- \cosh(\sigma) \left[ \frac{l_1 a_2}{\sigma^3} + \frac{l_1 a_2}{2} \left( \frac{1}{\sigma} + \frac{2}{\sigma^3} \right) + \frac{l_1 a_1}{\sigma^2} - \frac{a_1 b_{11}}{\sigma^2} \right. \\
 &\left. - \frac{a_2 b_{11}}{\sigma} \right] + \frac{a_1^2}{2\sigma^2} - \frac{a_2^2}{2\sigma^2} + \frac{l_1 b_{11}}{2} + \frac{l_1^2}{6} \left. \right\} \\
 Q_2 &= h Z_2 \left( \frac{p_2^2}{336} + \frac{a_3 p_2}{48} + \frac{7l_2 p_2}{120} + \frac{a_3^2}{30} + \frac{l_2 a_3}{6} \right. \\
 &\left. + \frac{b_{31} p_2}{8} + \frac{l_2^2}{6} + \frac{b_{31} a_3}{3} + \frac{b_{31} l_2}{2} \right)
 \end{aligned}$$

**A2. Case 1b: Diffusion of a Tracer with Lower Wall Impermeable (Insulating) and Upper Wall Permeable (Conducting) Wall Conditions**

$$\begin{aligned}
 b_1 &= Z_1 b_{11}; \quad b_2 = Z_1 b_{21} + Z_2 b_{22} + b_{2c} \\
 b_3 &= Z_1 b_{31} + Z_2 b_{32}; \quad b_4 = Z_1 b_{41} + Z_2 b_{42} + b_{4c} \\
 b_{11} &= \frac{a_1 \sinh(\sigma)}{\sigma} - \frac{a_2 \cosh(\sigma)}{\sigma} + l_1; \quad b_{41} = -b_{31} \\
 b_{42} &= - \left( \frac{p_2}{24} + \frac{a_3}{6} + \frac{l_2}{2} \right); \quad b_{4c} = 1 \\
 b_{31} &= \frac{-b_{41} \sinh(\alpha_2)}{\cosh(\alpha_2)}; \quad b_{32} = \frac{-b_{42} \alpha_2 \sinh(\alpha_2) - g_2}{\cosh(\alpha_2)} \\
 b_{3c} &= \frac{-b_{4c} \sinh(\alpha_2) + 1}{\cosh(\alpha_2)}; \quad b_{21} = b_{41} - \frac{a_1}{\sigma^2} \\
 b_{22} &= b_{42}; \quad b_{2c} = b_{4c}
 \end{aligned}$$

**A3. Case 2a: Diffusion of a Tracer for the Channel Filled with Porous Matrix (One-Fluid Model) with Impermeable Wall Conditions**

$$\begin{aligned}
 b_1 &= 0; \quad b_2 = 0; \\
 Q &= \frac{p^2 h^2}{DL} \frac{\partial C}{\partial \xi} \left[ -\tanh^2(\sigma) \left( \frac{2}{3\sigma^6} + \frac{4}{\sigma^8} \right) \right. \\
 &\left. + \frac{2}{\sigma^7} \tanh(\sigma) + \frac{1}{\sigma^6 \cosh^2(\sigma)} \left( 1 + \frac{\sinh(2\sigma)}{2\sigma} \right) \right]
 \end{aligned}$$

**A4. Case 2b: Diffusion of a Tracer for the Channel Filled with Porous Matrix (One-Fluid Model) with Lower Wall Impermeable and Upper Wall Permeable Wall Conditions**

$$b_1 = 0; \quad b_2 = \frac{h^2}{DL} \frac{\partial C}{\partial \xi} \frac{p}{\sigma^2} \left[ \frac{\tanh(\sigma)}{2\sigma} - \frac{1}{\sigma^2} \right] + 1$$

**A5. Case 3a: Diffusion of a Tracer in the Absence of Porous Matrix with Impermeable Wall Conditions (Two-Fluid Model)**

$$\begin{aligned}
 a_1 &= -a_4 + \frac{p_2}{2}; \quad a_2 = m n a_4; \quad a_3 = -a_4 - \frac{p_2}{2} \\
 a_4 &= \frac{-p_1}{2 m n (m + 1)} - \frac{-p_2 m}{2 (m + 1)} \\
 l c_1 &= -\frac{1}{2} \left( \frac{p_1}{6} - \frac{a_1}{2} - a_2 + \frac{p_2}{6} + \frac{a_3}{2} + a_4 \right) \\
 l c_2 &= -\frac{1}{2} \left( \frac{p_1}{6} - \frac{a_1}{2} + a_2 + \frac{p_2}{6} + \frac{a_3}{2} - a_4 \right) \\
 u_{1x} &= \frac{p_1 \eta^2}{2} + a_1 \eta + l c_1; \quad u_{2x} = \frac{p_2 \eta^2}{2} + a_3 \eta + l c_2 \\
 b_1 &= Z_1 b_{11}; \quad b_2 = 0; \quad b_3 = Z_2 b_{32} \\
 b_4 &= 0; \quad b_{11} = \frac{p_1}{6} - \frac{a_1}{2} + l c_1; \quad b_{32} = - \left( \frac{p_2}{6} + \frac{a_3}{2} + l c_2 \right)
 \end{aligned}$$

**A6. Case 3b: Diffusion of a Tracer in the Absence of Porous Matrix with Impermeable Wall Conditions (One-Fluid Model)**

$$A = \frac{p}{\alpha^3 \sinh(\alpha)}; \quad D^* = \frac{h^2 p^2}{D} \frac{\partial C}{\partial \xi} F(\alpha)$$