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MHD NATURAL CONVECTIVE FLOW OF A PARTICULATE SUSPENSION THROUGH A VERTICAL CHANNEL AT ASYMMETRIC THERMAL BOUNDARY CONDITIONS WITH HEAT GENERATION OR ABSORPTION

Ali J. Chamkha*
Manufacturing Engineering Department
The Public Authority for Applied Education and Training
Shuweikh, Kuwait

ABSTRACT

In this paper, a continuum model for two-phase (fluid/particle) flow induced by natural convection buoyancy effects is developed and applied to the problem of steady natural convection MHD flow of a two-phase particulate suspension through an infinitely long vertical channel in the presence of heat generation or absorption effects. The walls of the channel are heated asymmetrically such that one of the channel walls is maintained at a constant temperature while the other wall is maintained at constant heat flux. Particle-phase viscous effects are included in the model. While proper boundary conditions for a particle phase are unknown at present, boundary conditions borrowed from rarefied gas dynamics are employed for the particle-phase wall velocity conditions. Various closed-form solutions for different special cases of the problem are reported. A parametric study of some physical parameters involved in the problem is done to illustrate the influence of these parameters on the flow and thermal aspects of the problem.

Keywords: Two-phase flow; MHD; natural convection, channel; asymmetric conditions

NOMENCLATURE

$\vec{B}$ Magnetic induction [Wb m^{-2}]
c Fluid-phase specific heat at constant pressure [J $Kg^{-1}K$]
c_p Particle-phase specific heat at constant pressure [J $Kg^{-1}K$]

* E-mail: achamkha@yahoo.com

$\vec{g}$ Gravitational acceleration [m s^{-2}]
Gr Grashof number
H Channel width [m]
H Dimensionless buoyancy parameter
K Fluid-phase thermal conductivity [w $m^{-1}K^{-1}$]
M Hartmann number
N Interphase momentum transfer coefficient [s^{-1}]
N_T Interphase heat transfer coefficient [s^{-1}]
P Fluid-phase hydrostatic pressure [N m^{-2}]
Pr Fluid-phase Prandtl number
Q Heat generation/absorption coefficient [w m^{-3}]
q_2 Wall heat flux for T_1 - q_2 condition [J]
r_{tq} Walls thermal ratio for T_1 - q_2 condition
S Dimensionless particle-phase wall slip coefficient
T Fluid-phase temperature [K]
T_p Particle-phase temperature [K]
U Fluid-phase dimensionless velocity
u_p Particle-phase dimensionless velocity
U Fluid-phase velocity [ms^{-1}]
U_p Particle-phase velocity [ms^{-1}]
x,y Cartesian coordinates [m]

Greek Symbols

α Velocity inverse Stokes number
β Viscosity ratio
γ Specific heat ratio
ε Temperature inverse Stokes number
η Dimensionless y-coordinate
θ Dimensionless fluid-phase temperature
κ Particle loading
μ Fluid-phase dynamic viscosity [N s m^{-2}]
μ_p Particle-phase dynamic viscosity [N s m^{-2}]
Fluid-phase density [Kg m^{-3}]
ρ_p Particle-phase density [Kg m^{-3}]
σ Fluid-phase electrical conductivity [S m^{-1}]
Dimensionless heat generation/absorption coefficient
ω Particle-phase wall slip coefficient [m]

INTRODUCTION

Natural convection flow of a two-phase (particle/fluid) suspension is an important research area. This is due to the fact that this kind of flow is found in a wide range of applications including processes in the chemical and food industries, solar collectors where a particulate suspension is used to enhance absorption of radiation, cooling of electronic equipments, and cooling of nuclear reactors. In general, all applications of a single-phase flow are valid for a two-phase particulate suspension flow because the presence of contaminating particles in fluids occurs naturally. In spite of the importance, very little work has been done on natural convection of two-phase particulate suspensions. The evolution of cooling technology includes the progressive research of using natural convection, which is an inexpensive mode of heat transfer, in electronic equipments cooling. Vertical plates and channels are of the most encountered configurations used in natural convection cooling of electronic equipment.

A literature review in general for the historical papers reported in the development of cooling technology for electronic equipments has been presented by Bergles [1]. Later, an extensive review of electronic equipment cooling by different modes of heat transfer has been presented by Incorpora [2]. The review includes natural convection heat transfer in parallel channels, inclined channels and enclosures as well as other configurations with different operating conditions. The importance of heat transfer considerations in the design of electronic equipment has been studied extensively and reported by Aung and Chaimah [3], Jaluria [4], Kraus and Bar-Cohen [5], and Stienberg [6].

Akbari and Borgers [7] studied free convection laminar heat transfer between the channel surfaces of the trombe wall. The study was done using a line-by-line forward marching implicit finite-difference technique. The study was restricted to laminar flow between two parallel plates, each at some effective uniform temperature. Yao [8] investigated the problem of mixed convection in vertical channel. An analytical solution is developed to study the hydrodynamically and thermally developing laminar flow in a heated channel. The transient effects in natural convection cooling of vertical parallel plates are reported by Joshi [9]. Aung [10] considered fully developed laminar free convection between vertical plates heated asymmetrically. Aung et al. [11] reported on the development of laminar free convection between vertical flat plates with asymmetric heating. A more detailed reference list was given by Muhanna [12] who investigated numerically laminar natural convection flows in obstructed vertical channels. Aung and Worku [13, 14] discussed the theory of combined free and forced convection in a vertical channel with flow reversal conditions for both developing and fully developed flows. Aung and Worku [14] assumed that the walls of the channel were having asymmetric temperatures. The case of developing mixed convection flow in ducts with asymmetric wall heat fluxes was analyzed by the same authors [15]. Chamkha [16] reported on laminar hydromagnetic mixed convection flow in a vertical channel with symmetric and asymmetric wall heating conditions for a single-phase flow. Related references for natural and mixed convection flows of a single phase are given in the book by Gebhart et al. [17].

Various aspects of forced convection two-phase particulate suspension flows through a parallel-plates channel have been considered by many previous investigators. For example, Ritter [18] reported transient two-phase fluid-particle flows in channels and circular pipes. He found that the presence of particles caused significant reductions in the flow rates of both phases. Chamkha [19, 20] reported analytical solutions for transient hydromagnetic two-phase flow in a channel with constant and oscillating pressure gradients. Chamkha [21] also solved analytically the problem of unsteady laminar hydromagnetic fluid-particle flow and heat transfer in channels and circular pipes under oscillating and ramp applied pressure gradients. Attia [22] considered unsteady hydromagnetic channel flow of dusty fluid with temperature dependent viscosity and thermal conductivity.

On the other hand, very little work has been reported on natural convection flow of a particle-fluid suspension over and through different geometries. Chamkha and Ramadan [23] and Ramadan and Chamkha [24] have reported some analytical and numerical results for natural convection flow of a two-phase particulate suspension over an infinite vertical plate. Also, Okada and Suzuki [25] have considered buoyancy-induced flow of a two-phase suspension in an enclosure. Al-Subaie and Chamkha [26] performed an analytical study dealing with natural convection flow of a particulate suspension through a vertical channel with isothermal walls. Some research have been done concerning the thermophoretic transport involved in viscous flow and in flow in porous media for various geometries and boundary conditions. A comprehensive review of the pertinent literature has recently been given by Grosan et al. [27], Selim et al. [28], Postelnicu [29] and Chamkha and Pop [30]. Recently, Magyari [31] considered thermophoretic deposition of particles in fully developed mixed convection flow in a parallel-plate vertical channel and reported a full analytical solution. Chamkha and Al-Rashidi [32] reported analytical solutions for hydromagnetic natural convection flow of a particulate suspension through isoflux-isothermal channels in the presence of a heat source or sink.

The present work includes particle-phase viscous stresses which can be used to model particle-particle interaction. They can also be thought of as a natural consequence of the averaging processes employed to model a discrete system of particles as a continuum (see, for instance, Drew and Segal [33] and Drew [34]). Previously published work which includes the particle-phase viscous stresses can be found in the papers by Gidaspow [35], Tsuo and Gidaspow [36], and Gadiraju et al. [37]. In the present work, both the fluid- and the particle-phases are assumed to be incompressible and have constant properties except the density in the buoyancy term of the fluid-phase momentum equation, and the particle-phase volume fraction is assumed to be constant and finite. In reality, the particle-phase volume fraction is non-uniform but it is assumed to be constant herein so as to allow the governing equations to be solved analytically. In order to model many situations, multiphase (fluid-particle) theories will eventually need to allow for a transition of the particle-phase from fluid-like behavior at low volume fraction to solid-like behavior at large volume fraction. The inclusion of a particle phase viscosity as a function of the volume fraction represents a step in that direction (Gadiraju et al. [37]). It should be noted that if the particle-phase viscosity is allowed to increase rapidly with the volume fraction, the particle-phase will behave as a rigid body at large values of the volume fraction. This is not done herein and the particle-phase viscosity is assumed to be constant in order to obtain closed-form solutions. However, there have been some published works which predict variable particle-phase volume fraction (see, for instance, Soo [38], Sinclair and Jackson [39] and Drew and Lahey [40]). The inclusion of a lift force in the governing equations is sufficient (but not necessary) to produce non-uniform volume fraction distribution. In addition, as a result of the inclusion of the particle-phase viscous stresses, the particle-phase will have a corresponding pressure. This pressure will be constant for uniform particle-phase volume fraction situations and, therefore, the particle-phase pressure gradient will vanish.

The objective of the present work is to study natural convection laminar flow of a particulate suspension in isoflux-isothermal vertical channels in the presence of particle-phase viscous stresses, magnetic field and heat generation or absorption effects.

PROBLEM FORMULATION

Consider steady, laminar, natural convection fully-developed flow of a particulate (fluid/particle) suspension in a vertical parallel-plate channel. The channel walls are heated asymmetrically (isoflux-isothermal walls) such that the left wall of the channel is maintained at a constant heat flux q_1 while the right wall is kept at a constant temperature. The schematic of the problem is shown in Figure 1. The fluid phase is assumed to be Newtonian, viscous, electrically conducting, and heat generating or absorbing. The particle phase is assumed to be made up of discrete particles of one size and constant density. The particle phase is assumed to be pressureless and electrically non-conducting. Both phases are assumed to be interacting continua and the particle volume fraction is assumed to be small (Marble, [41]). The governing equations for this investigation are based on the balance laws of mass, linear momentum and energy for both the fluid and particle phases. They can be written in vector form as

$$\vec{\nabla}\cdot(\rho\vec{V}) = 0 \tag{1}$$

$$\rho\vec{V}.\vec{\nabla}\vec{V} = -\vec{\nabla}P + \vec{\nabla}.(\mu\vec{\nabla}\vec{V}) - \rho_p N(\vec{V} - \vec{V}_p) + \rho\vec{g} + \sigma(\vec{V}\times\vec{B})\times\vec{B} \tag{2}$$

$$\rho c\vec{V}.\vec{\nabla}T = \vec{\nabla}.(k\vec{\nabla}T) + \rho_p c_p N_T(T_p - T) \pm Q(T - T_\circ) \tag{3}$$

$$\vec{\nabla}.(\rho_p V_p) = 0 \tag{4}$$

$$\rho_p\vec{V}_p.\vec{\nabla}\vec{V}_p = \vec{\nabla}.(\mu_p\vec{\nabla}\vec{V}_p) + \rho_p N(\vec{V} - \vec{V}_p) + \rho_p\vec{g} \tag{5}$$

$$\rho_p c_p\vec{V}.\vec{\nabla}T_p = -\rho_p c_p N_T(T_p - T) \tag{6}$$

where $\vec{V}$ and $\vec{V}_p$ are the velocity vectors of the fluid and particle phases, respectively. T and T_p are the temperatures of the fluid and particle phases, respectively. $\vec{g}$ is the gravity vector, $\vec{\nabla}P$ is the pressure gradient vector, $T_\circ$ is the temperature at a reference point "$\circ$" in the channel, $\vec{B}$ is the magnetic induction vector and Q is the heat generation or absorption coefficient depending on its sign. The other parameters, namely, ρ, μ, σ, c and k are the density, dynamic viscosity, electrical conductivity, specific heat and thermal conductivity of the fluid phase, while ρ_p, μ_p, c_p are the particle-phase density, dynamic viscosity and specific heat. It should be mentioned that in the previous equations, the terms containing the momentum transfer coefficient N represent the interphase drag between the phases while the terms containing the heat transfer coefficient N_T represent the inter-phase heat transfer between the fluid and the particle phases.

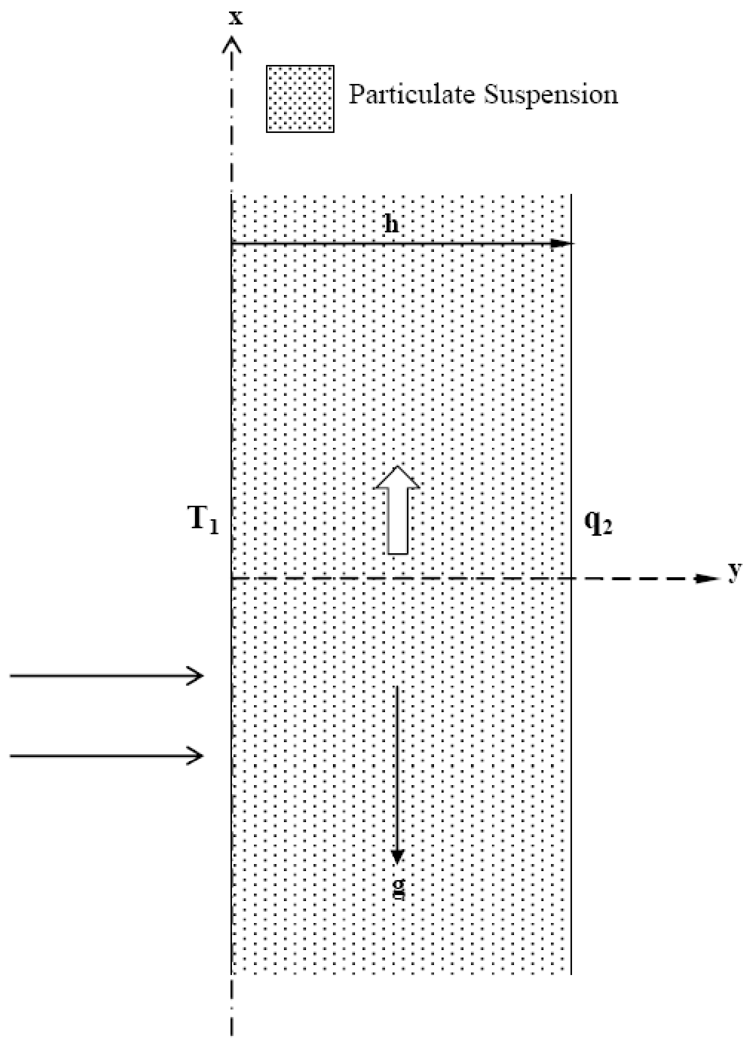

Figure 1. Schematic of the problem.

Following the same procedure done by Al-Subaie and Chamkha [26] by evaluating the governing equations at a reference point "$\circ$" at the entrance of the channel such that $\vec{V} = 0, T = T_\circ, \rho = \rho_\circ, \mu = \mu_\circ, \sigma = \sigma_\circ, \vec{V}_p = \vec{V}_{p_\circ}, T_p = T_{p_\circ}, \rho_p = \rho_{p_\circ}$ and $\mu_p = \mu_{p_\circ}$ and employing the Boussinesq approximation, Equation (2) becomes

$$\vec{V}.\vec{\nabla}\vec{V} = -\frac{\rho_{p_\circ}}{\rho_\circ}\vec{g} + \frac{\mu_\circ}{\rho_\circ}\vec{\nabla}.\vec{\nabla}\vec{V} - \frac{\rho_{p_\circ}}{\rho_\circ}N(\vec{V}-\vec{V}_p) - \beta^*(T-T_\circ)\vec{g} + \frac{\sigma_\circ}{\rho_{p_\circ}}(\vec{V}\times\vec{B})\times\vec{B} \tag{7}$$

where β^* is the volumetric expansion coefficient.

Since the walls of the channel are assumed to be infinitely long, the dependence of the variables on the x-direction will be negligible compared with that of the y-direction. Therefore, all dependent variables in the governing equations will only be functions of y as follows:

$$\vec{V} = U(y)\vec{e}_x, \quad \vec{V}_p = U_p(y)\vec{e}_x, \; T = T(y) \quad , \quad T_p = T_p(y) \tag{8a-d}$$

where $U(y)$ is the fluid phase x-component of velocity, $U_p(y)$ is the particle-phase x-component of velocity, $T(y)$ and $T_p(y)$ are the fluid phase and particle phase temperatures, respectively and $\vec{e}_x$ is the unit vector in the x-direction. These assumptions also imply that the constant vector $\vec{g}$ will be reduced to $g\vec{e}_x$ which is the magnitude of the acceleration due to gravity component in the x- direction. Also, assuming that the fluid is electrically-conducting and is subjected to a uniform transverse magnetic field which is applied normally to the flow direction (see Figure 1), the electromotive force $\sigma\vec{V}\times\vec{B}$ in Equation (7) will provide a current whose interaction with $\vec{B}$ will decelerate the flow. This implies that

$$\sigma(\vec{V}\times\vec{B})\times\vec{B} = -\sigma B^2 U(y)\vec{e}_x \tag{9}$$

where $\vec{B}$ is the magnitude of magnetic induction. Taking all of the above assumptions into consideration, the governing equations reduce to

$$\mu_\circ d_{yy}U - \rho_{p_\circ}N(U-U_p) - \sigma_\circ B^2 U + \beta^*\rho_\circ g(T-T_\circ) + \rho_{p_\circ}g = 0 \tag{10}$$

$$k d_{yy}T + \rho_p c_p N_T(T_p - T) \pm Q(T-T_\circ) = 0 \tag{11}$$

$$\mu_p d_{yy}U_p + \rho_p N(U-U_p) - \rho_p g = 0 \tag{12}$$

$$T_p - T = 0 \tag{13}$$

It should be noted that the continuity equations of both phases are identically satisfied.

The physical boundary conditions for this problem are:

$$U(0) = U(h) = 0, \; T(0) = T_1, \; \frac{dT}{dy}(h) = -\frac{q_2}{k} \tag{14a-d}$$

$$U_p(0) = \omega\, d_y U_p(0) - \frac{g}{N},\ U_p(h) = -\,\omega\, d_y U_p(h) - \frac{g}{N} \qquad (14e,f)$$

where h is the channel width, T_1 is the channel temperature at y = 0, q_2 is the wall heat flux at y=h , and ω is the particle-phase wall slip coefficient. Equations (14a) and (14b) indicate no slip conditions for the fluid-phase at the wall of the channel. Equations (14c) and (14d) suggest that the left wall is kept at constant temperature T_1 while the right wall of the channel is kept at a constant heat flux q_2.

Equations (14e) and (14f) express proposed wall boundary conditions for the particle phase at the left and right walls of the channel. It should be mentioned herein that the wall boundary conditions for the particulate phase are poorly understood at this time. However, there is experimental evidence that particles tend to slip at a boundary. Therefore, two idealized conditions can be considered. These are the no slip condition (ω=0) and the perfect slip condition [$\omega\to\infty$: $d_yU_p(0)= d_yU_p(h) = 0$]. It is expected that the actual behavior would be somewhere between these two extremes. The term g/N appears in the boundary conditions for the particle phase accounting for the gravitational buoyancy.

It is convenient to non-dimensionalize the governing equations and conditions. This can be accomplished by using the following parameters:

$$y = h\eta,\quad U = \frac{\mu}{\rho h}u,\quad U_p = \frac{\mu}{\rho h}u_p,\quad T = \frac{q_2 h}{k}\theta + T_\circ,\quad T_p = \frac{q_2 h}{k}\theta_p + T_\circ \qquad (15)$$

where η is the dimensionless transverse coordinate, u and u_p are the dimensionless fluid and particle-phase velocities, respectively. θ and θ_p are the dimensionless fluid-phase and particle-phase temperatures, respectively.

After performing the mathematical operations, the resulting dimensionless governing equations can be written as:

$$d_{\eta\eta}u - \kappa\alpha\,(u - u_p) - M^2 u + Gr\,\theta + \kappa H = 0 \qquad (16)$$

$$\frac{1}{Pr}\, d_{\eta\eta}\theta + \kappa\,\gamma\,\varepsilon\,(\theta_p - \theta) \pm \phi\,\theta = 0 \qquad (17)$$

$$\beta\, d_{\eta\eta}u_p + \alpha\,(u - u_p) - H = 0 \qquad (18)$$

$$\theta_p - \theta = 0 \qquad (19)$$

where

$$\alpha = h^2\,\frac{N\rho}{\mu},\ \kappa = \frac{\rho_p}{\rho},\ Gr = \frac{\beta^* q_2 h^4 \rho^2 g}{\kappa\mu^2},\ M^2 = \frac{\sigma B^2 h^2}{\mu},\ H = \frac{\rho^2 h^3 g}{\mu^2},\ Pr = \frac{\mu c}{\kappa},$$

$$\gamma = \frac{c_p}{c},\ \varepsilon = \frac{\rho N_T h^2}{\mu},\ \phi = \frac{Qh^2}{\mu c},\ \beta = \frac{\nu_p}{\nu} \tag{20}$$

are the momentum inverse Stokes number, the particle loading, the Grashof number, square of Hartmann number, buoyancy parameter, the Prandtl number, the specific heat ratio, the temperature inverse Stokes number, the heat generation or absorption parameter, and the viscosity ratio, respectively.

The dimensionless boundary conditions become

$$u(0) = u(1) = 0,\ \theta(0) = r_{tq},\ \left.\frac{d\theta}{d\eta}\right|_{\eta=1} = -1 \tag{21a-d}$$

$$u_p(0) = S\, d_\eta u_p(0) - \frac{H}{\alpha},\ u_p(1) = -S\, d_\eta u_p(1) - \frac{H}{\alpha} \tag{21e,f}$$

where $S=\frac{\omega}{h}$ is the dimensionless particle-phase wall slip parameter and $r_{tq}=\frac{T_1 - T_\circ}{q_2 h/k}$ is the walls thermal ratio. It should be mentioned that when $\beta = 0$ (inviscid particle phase), Equations (21e,f) are ignored.

ANALYTICAL SOLUTIONS

In this section, analytical solutions for various special cases of the problem under consideration in the presence ($\beta \neq 0$) or absence ($\beta=0$) of particle-phase viscous stresses are reported.

Case 1: This case considers steady natural convection two-phase flow through an isothermal-isoflux vertical channel in the absence of magnetic field ($M = 0$) and heat generation or absorption $(\phi = 0)$ for an inviscid particle phase $(\beta = 0)$. The governing equations for this case can be written as follows:

$$D^2 u - \alpha\kappa\,(u - u_p) + Gr\,\theta + \kappa H = 0 \tag{22}$$

$$\frac{1}{Pr} D^2\theta + \kappa\gamma\varepsilon\,(\theta_p - \theta) = 0 \tag{23}$$

$$\alpha\,(u - u_p) - H = 0 \tag{24}$$

$$\varepsilon\,(\theta_p - \theta) = 0 \tag{25}$$

where the D^2 denotes a second derivative operator with respect to η.

The boundary conditions are:

$u(0) = u(1) = 0$ (26a,b) $\theta(0) = r_{tq}$, $\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1} = -1$ (26c,d)

Equation (24) implies that

$$u_p(\eta) = u(\eta) - H/\alpha \tag{27}$$

Which indicates that the particle-phase velocity is the same as the fluid-phase velocity except that it is shifted by the factor $\frac{H}{\alpha}$ below the fluid-phase velocity. Equation (25) implies that

$$\theta_p(\eta) = \theta(\eta) \tag{28}$$

By substituting equation (28) into equation (23) one obtains

$$D^2\theta = 0 \tag{29}$$

The solution of this simple second-order differential equation, which satisfies the boundary conditions (26 c,d) is

$$\theta(\eta) = -\eta + r_{tq} \tag{30}$$

This indicates that the temperature of both phases has a linear shape of pure conduction.

Now, substituting equations (30) and (27) into equation (22) gives

$$D^2u = G_r(-\eta + r_{tq}) \tag{31}$$

The solution of this second-order differential equation subject to the boundary conditions (26 a,b) is

$$u(\eta) = \frac{Gr(\eta(\eta^2 - 1) - 3\eta(\eta - 1)r_{tq})}{6} \tag{32}$$

This shows that the fluid-phase velocity profile has a cubic relation with the normal distance. The corresponding solution for $u_p(\eta)$ is obtained by substituting equation (32) into equation (27) to get

$$u(\eta) = \frac{Gr(\eta(\eta^2 - 1) - 3\eta(\eta - 1)r_{tq})}{6} - \frac{H}{\alpha} \tag{33}$$

Case 2: This case considers steady natural convection two-phase flow through an isothermal-isoflux vertical channel subjected to a uniform transverse magnetic field which is applied normally to the flow direction and the flow has neither heat generation or absorption effects $(\phi = 0)$ nor particle- phase viscosity $(\beta = 0)$. The governing equations and the boundary conditions will be the same as Case1 except the fluid-phase momentum equation, which can be written as

$$D^2u - \alpha\kappa(u - u_p) + Gr\theta - M^2u + \kappa H = 0 \tag{34}$$

Substituting equations (27) and (30) into equation (34) gives

$$D^2u - M^2u = -\mathrm{Gr}\,(-\eta + r_{tq}) \tag{35}$$

Solving the above equation subject to (26a,b) gives the following fluid-phase velocity:

$$u(\eta) = c_1 e^{M\eta} + c_2 e^{-M\eta} + \frac{Gr(r_{tq} - \eta)}{M^2} \tag{36}$$

where

$$c_1 = \frac{-Gr(r_{tq}e^M + 1 - r_{tq})/M^2}{e^{-M} - e^M} - \frac{Gr}{M^2} r_{tq} \tag{37}$$

$$c_2 = \frac{Gr(r_{tq}e^M + 1 - r_{tq})/M^2}{e^{-M} - e^M} \tag{38}$$

The corresponding particle-phase velocity profile $u_p(\eta)$ can be written as

$$u_p(\eta) = c_1 e^{M\eta} + c_2 e^{-M\eta} + \frac{Gr(r_{tq} - \eta)}{M^2} - \frac{H}{\alpha} \tag{39}$$

Figure 2 shows the linear relationship between the fluid-phase and particle-phase temperature profiles which have decreasing trends from the first wall to the second wall for all values of r_{tq}. The temperature distribution of both phases increase as the value of r_{tq} increases.

Figures 3 and 4 present typical velocity profiles for the fluid and particle phases for different values of r_{tq}, respectively. It is interesting to notice that unlike the case of isoflux-isothermal thermal condition, the fluid-phase velocity profile shows that for moderate values of r_{tq} (0.5, 0.75) it increases near the isoflux wall and decreases close to the isothermal wall. This behavior is different for $r_{tq} = 1$ where no back flow occurs in the entire channel. Also, for $r_{tq} = 0.1$, the flow is going downward since the temperature field is almost all negative for this value of r_{tq}. Similar velocity profiles are predicted for the particle phase except u_p is negative for all values of r_{tq} since the particles are free falling due to gravity effects.

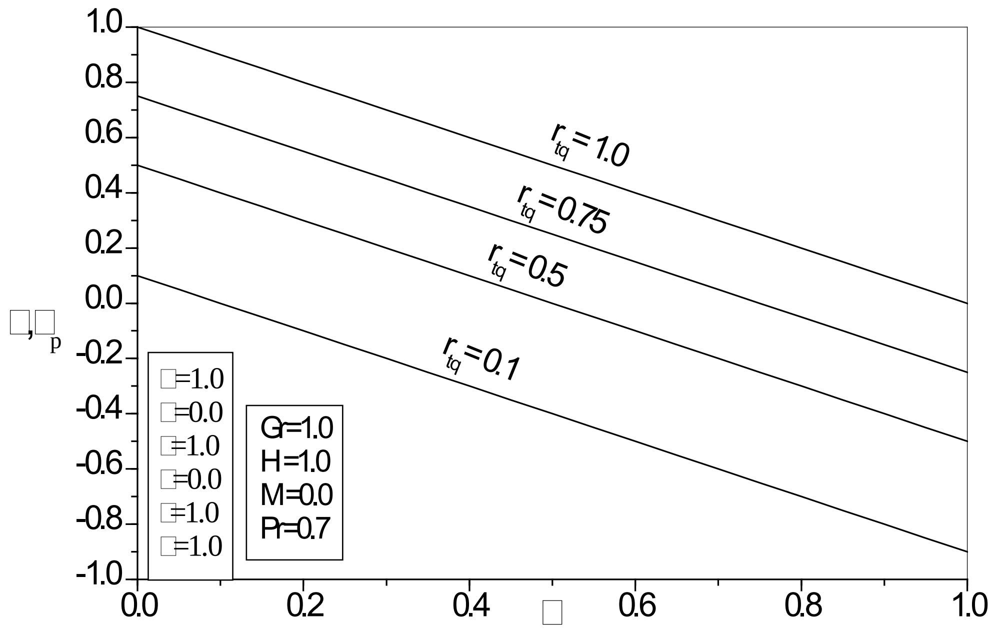

Figure 2. Effects of r_{tq} on fluid and practicle phase temperature profiles.

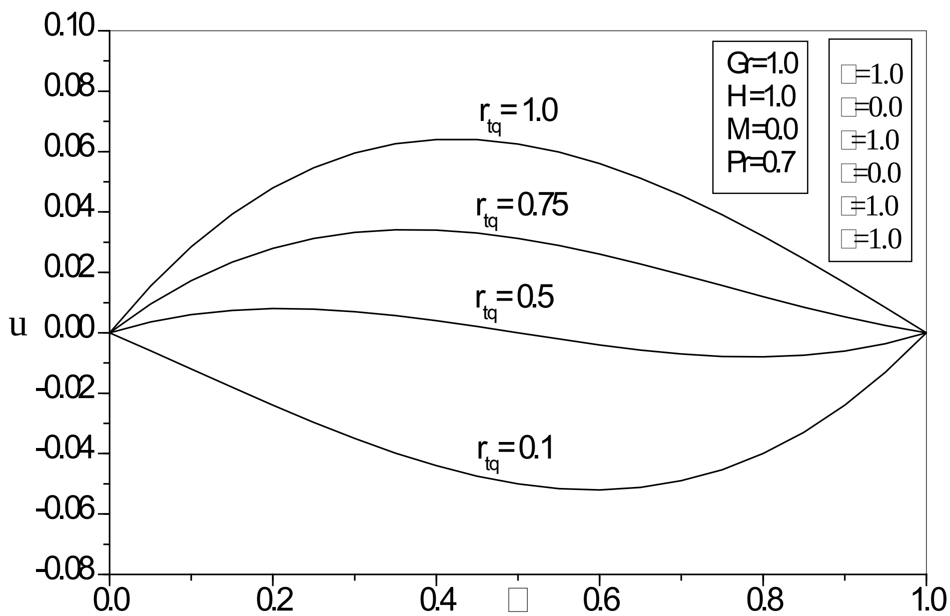

Figure 3. Effects of r_{tq} on fluid phase velocity profile.

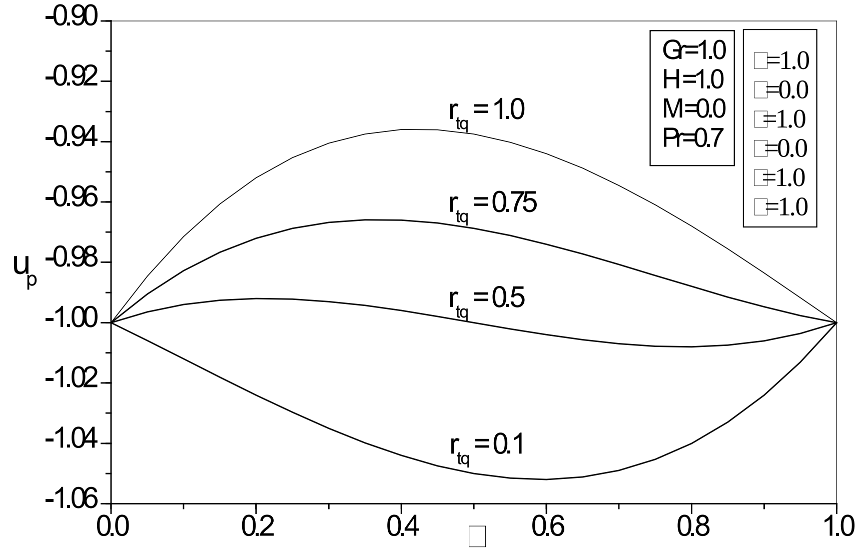

Figure 4. Effects of r_{tq} on particle phase velocity profile.

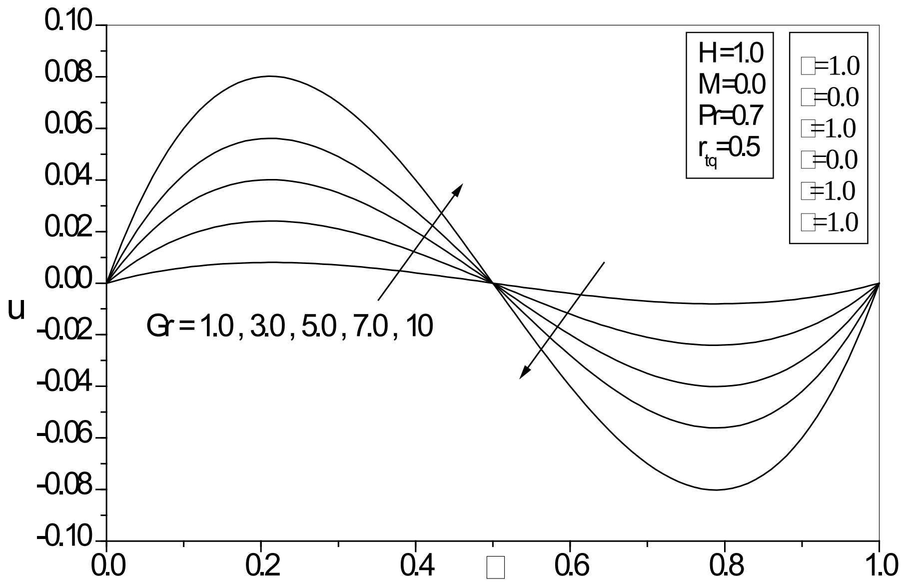

Figure 5. Effects of Gr on fluid phase velocity profile.

Figures 5 and 6 display the effects of increasing the Grashof number Gr on the velocity fields of both the fluid and particle phases, respectively. Increasing the value of Gr increases the thermal buoyancy effect. This increases the upward flow of both phases close to the isothermal wall and backward flow near the isothermal wall.

Figures 7 and 8 present velocity profiles for the fluid and particle phases for various values of the Hartmann number M, respectively. As M increases, the velocities of both phases decrease because application of a transverse magnetic field produces a drag-like force called the Lorentz force which acts in the direction opposite to that of motion.

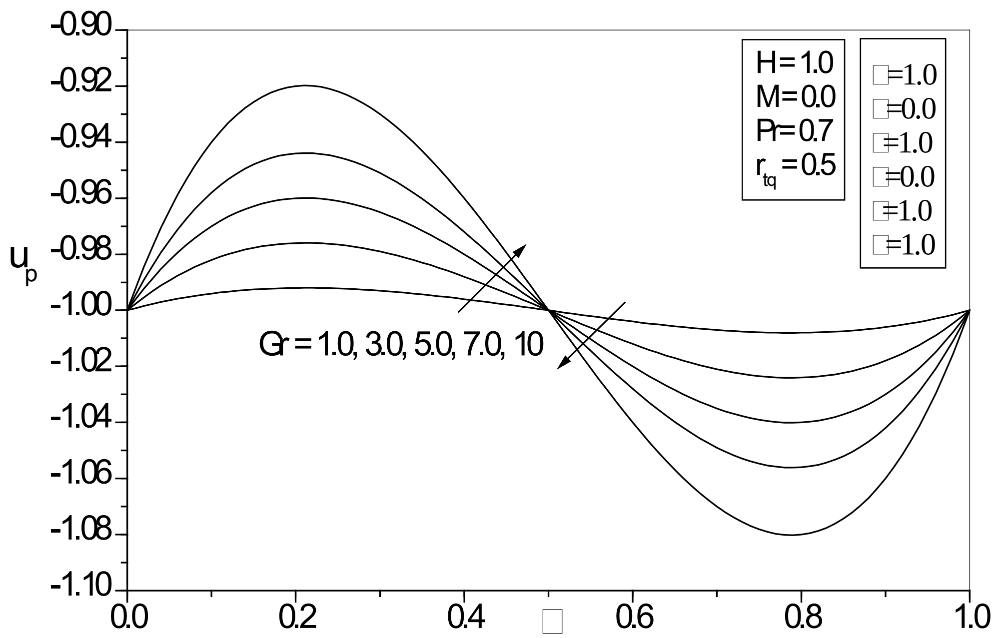

Figure 6. Effects of Gr on particle phase velocity profile.

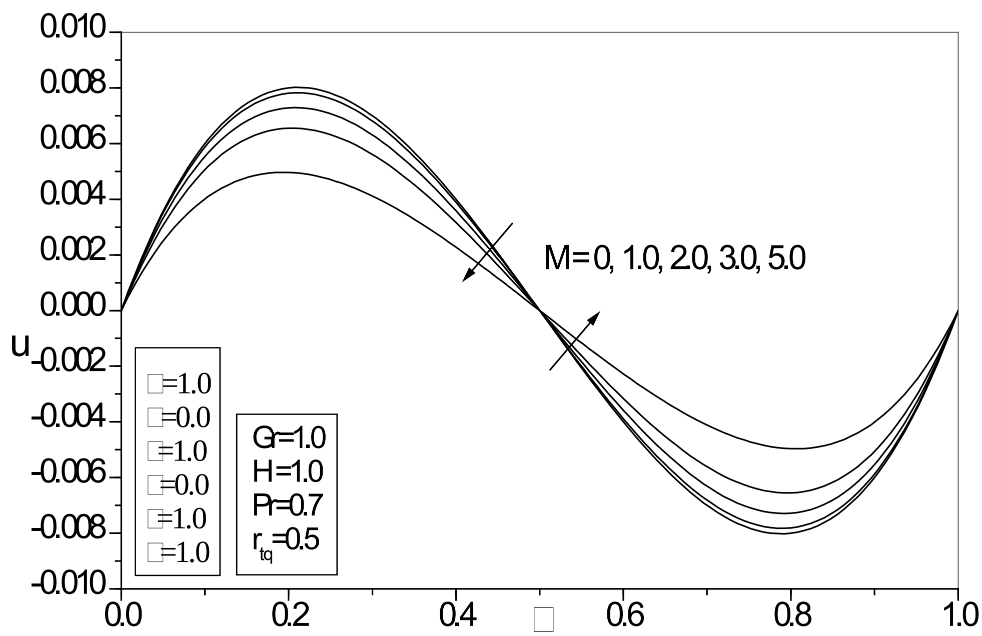

Figure 7. Effects of M on fluid phase velocity profile.

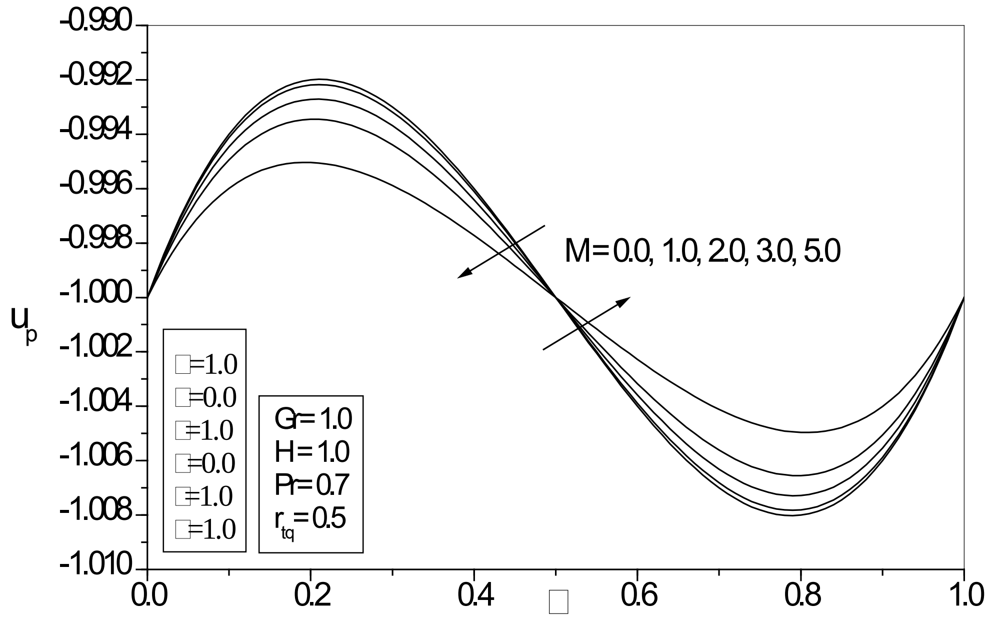

Figure 8. Effects of M on particle phase velocity profile.

Case 3: This case considers teady natural convection two-phase flow through an isothermal-isoflux vertical channel in the presence of a heat source $(\phi > 0)$ and a magnetic field, without particle phase viscosity $(\beta = 0)$. The governing equations and the boundary conditions for this case will be the same as Case 2 except the fluid-phase energy equation, which can be written as follows

$$\frac{1}{\Pr} D^2\theta + \kappa\gamma\varepsilon\left(\theta_p - \theta\right) - \phi\theta = 0 \tag{40}$$

Substituting equation (28) into equation (40) gives:

$$\frac{1}{\Pr} D^2\theta - \phi\theta = 0 \tag{41}$$

Solving the above differential equation subject to (26c,d) gives the following fluid-phase temperature:

$$\theta(\eta) = \theta_p(\eta) = \left(\frac{-1 - r_{tq}\sqrt{\phi \Pr}\ \ \sinh\sqrt{\phi \Pr}}{\sqrt{\phi \Pr}\ \cosh\sqrt{\phi \Pr}} \right) \sinh\sqrt{\phi \Pr}\ \ \eta$$
$$+\ r_{tq}\ \cosh\sqrt{\phi \Pr}\ \ \eta \tag{42}$$

Substituting equations (27)and (42) into (34) yields

$$D^2u - M^2u = -Gr\theta(\eta) \tag{43}$$

Solving this differential equation subject to the boundary conditions (26a,b) gives the following fluid-phase velocity:

$$u(\eta) =$$

$$\frac{-Gr\, r_{tq}}{\phi \Pr - M^2}\left[\cosh\sqrt{\phi \Pr}\eta - \cosh M\eta + \frac{\cosh M \sinh M\eta}{\sinh M} - \frac{\cosh\sqrt{\phi \Pr}\sinh M\eta}{\sinh M}\right]$$

$$+\mathrm{Gr}\left[\frac{1 + r_{tq}\sqrt{\phi \Pr}\sinh\sqrt{\phi \Pr}}{(\phi \Pr - M^2)\sqrt{\phi \Pr}\cosh\sqrt{\phi \Pr}}\right]\left[\sinh\sqrt{\phi \Pr}\eta - \frac{\sinh\sqrt{\phi \Pr}\sinh M\eta}{\sinh M\eta}\right] \tag{44}$$

The particle-phase velocity profile is obtained by substituting equation (44) into equation (27).

This solution can be used to obtain the corresponding problem without a magnetic field (M = 0). Thus, the fluid-phase velocity of such problem will be obtained by setting M equals to zero and applying L'Hospital's rule to equation (44). If this is done, one obtains:

$$u(\eta) = -\frac{Gr\, r_{tq}}{\phi \Pr}\left(\cosh\sqrt{\phi \Pr}\,\eta - \eta\cosh\sqrt{\phi \Pr} + \eta - 1\right)$$

$$+\frac{Gr\left(1 + r_{tq}\sqrt{\phi \Pr}\sinh\sqrt{\phi \Pr}\right)}{(\phi \Pr)^{3/2}\cosh\sqrt{\phi \Pr}}\left(\sinh\sqrt{\phi \Pr}\eta - \eta\sinh\sqrt{\phi \Pr}\right) \tag{45}$$

Case 4: This case considers steady natural convection two-phase flow through an isothermal-isoflux vertical channel in the presence of a heat sink $(\phi < 0)$ and a magnetic field, without particle phase viscosity $(\beta = 0)$. The governing equations and boundary conditions will be the same as in Case 2 except the fluid-phase energy equation, which can be written as:

$$\frac{1}{\Pr}D^2\theta + \kappa\gamma\varepsilon(\theta_p - \theta) + \phi\theta = 0 \tag{46}$$

Substituting equation (28) into equation (44) gives:

$$\frac{1}{\Pr}D^2\theta + \phi\theta = 0 \tag{47}$$

The solution of equation (47) subject to (26c,d) gives the following fluid-phase temperature:

$$\theta(\eta) = r_{tq}\cos\sqrt{\phi \Pr}\eta - \left(\frac{1 + r_{tq}\sqrt{\phi \Pr}\sin\sqrt{\phi \Pr}}{\sqrt{\phi \Pr}\cos\sqrt{\phi \Pr}}\right)\sin\sqrt{\phi \Pr}\eta \tag{48}$$

By substituting equations (27)and (46) into equation (34) and rearranging one obtains

$$D^2u - M^2u = -\,Gr\,\theta(\eta) \tag{49}$$

Equation (49) can be solved subject to the flow boundary conditions given in equations (26a,b) by the usual method of solving such equations to give the following fluid-phase velocity:

$$u(\eta) = \frac{Gr \quad r_{qt}}{\phi \Pr + M^2}\left[\frac{\sinh M(\eta - 1)}{\sinh M} + \cos\sqrt{\phi \Pr}\,\eta - \frac{\cos\sqrt{\phi \Pr}\ \sinh M\eta}{\sinh M}\right]$$
$$-\,Gr\left[\frac{1 + r_{tq}\sqrt{\phi \Pr}\sin\sqrt{\phi \Pr}}{(\phi \Pr + M^2)\sqrt{\phi \Pr}\ \cos\sqrt{\phi \Pr}}\right]\left[\sin\sqrt{\phi \Pr}\,\eta - \frac{\sin\sqrt{\phi \Pr}\sinh M\eta}{\sinh M}\right] \tag{50}$$

Again, this solution can be used to solve the corresponding problem in the absence of a magnetic field. Setting M equals to zero and applying L'Hospital's rule to equation (50) will give the fluid-phase velocity for this problem as:

$$u(\eta) = \frac{Gr\, r_{tq}}{\phi \Pr}\left(\cos\sqrt{\phi \Pr}\,\eta - \eta\cos\sqrt{\phi \Pr} + \eta - 1\right)$$
$$-Gr\left(\sin\sqrt{\phi \Pr}\eta - \eta\sin\sqrt{\phi \Pr}\right)\left[\frac{1 + r_{tq}\sqrt{\phi \Pr}\sin\sqrt{\phi \Pr}}{(\phi \Pr)^{3/2}\cos\sqrt{\phi \Pr}}\right] \tag{51}$$

Some results for θ, θ_p, u and u_p based on the closed-form solutions for the flow through a vertical channel in the presence of a heat generation (source) or a heat absorption (sink) term $\pm\,\phi$ are presented in Figures 9 through 14 . Figures 9 shows that θ and θ_p increase as ϕ decreases. Figure 10 shows that both θ and θ_p increase as Pr increases for ϕ = -1 and they decrease as Pr increases for ϕ = 1.

Figures 11 and 12 show representative velocity profiles for both phases (u and u_p) for different values of the heat generation or absorption coefficient ϕ. For the isothermal-isoflux condition increases in the values of ϕ have a tendency to decrease the buoyancy effects close to the hot wall.

This produces a reduction in the fluid-phase and particle-phase velocities there as clearly depicted in Figures 11 and 12. However, it is also observed that for $\phi = -0.5$ and -1.0 no back flow occurs near the isoflux wall.

The effects of the Prandtl number Pr on the velocity profiles of both phases for heat generation or absorption conditions are shown in Figures 13 and 14, respectively. It is observed that for heat generation ($\phi > 0$) as Pr increases, the velocities of both phases decrease while they increase in the presence of heat absorption effects.

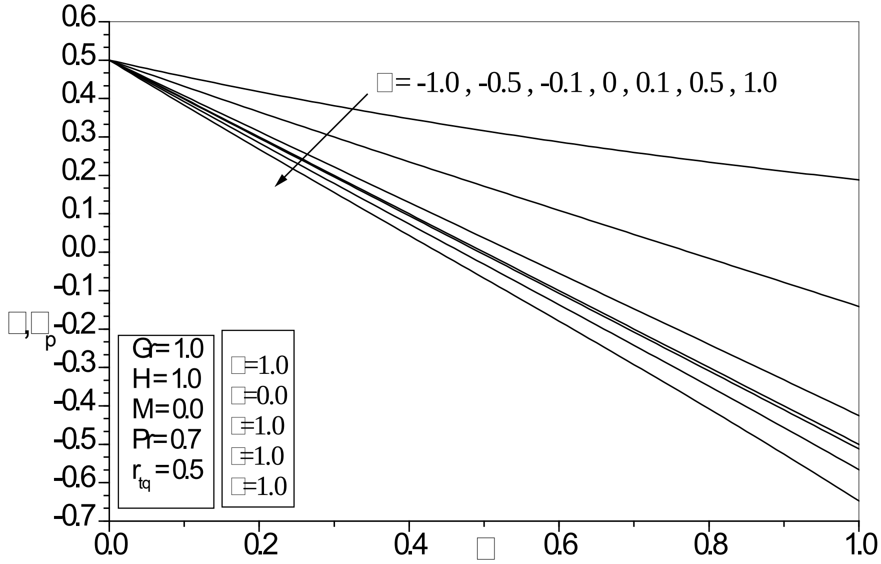

Figure 9. Effects of ϕ on fluid and particle phase temperature profiles.

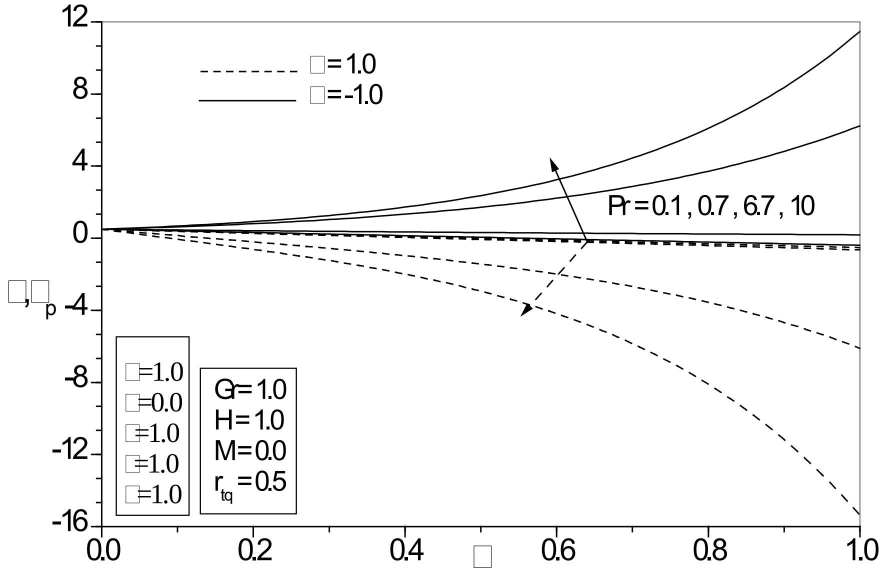

Figure 10. Effects of P_r on fluid and particle phase temperature profiles.

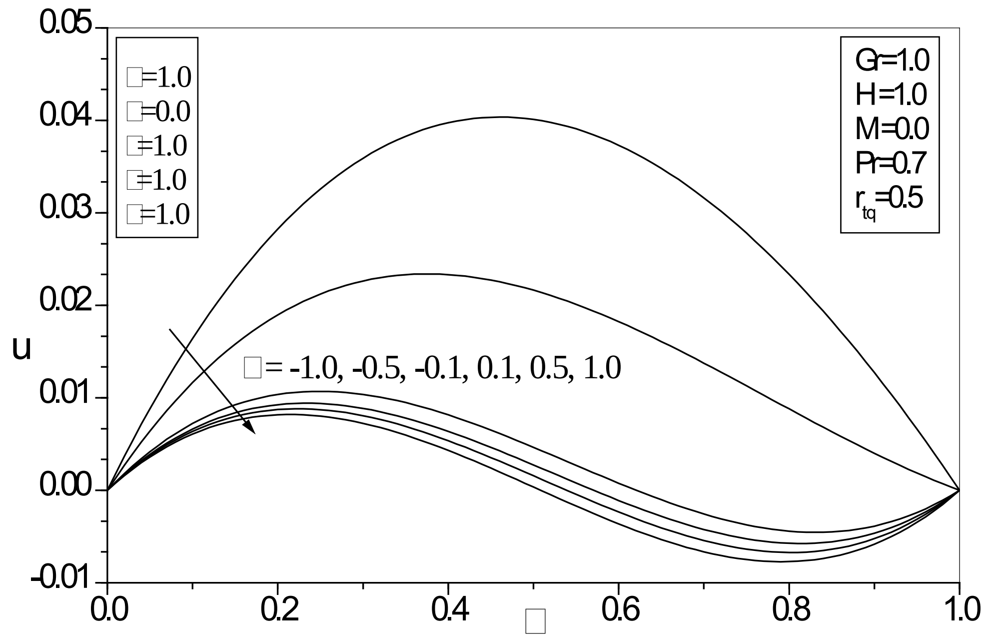

Figure 11. Effects of ϕ on fluid phase velocity profile.

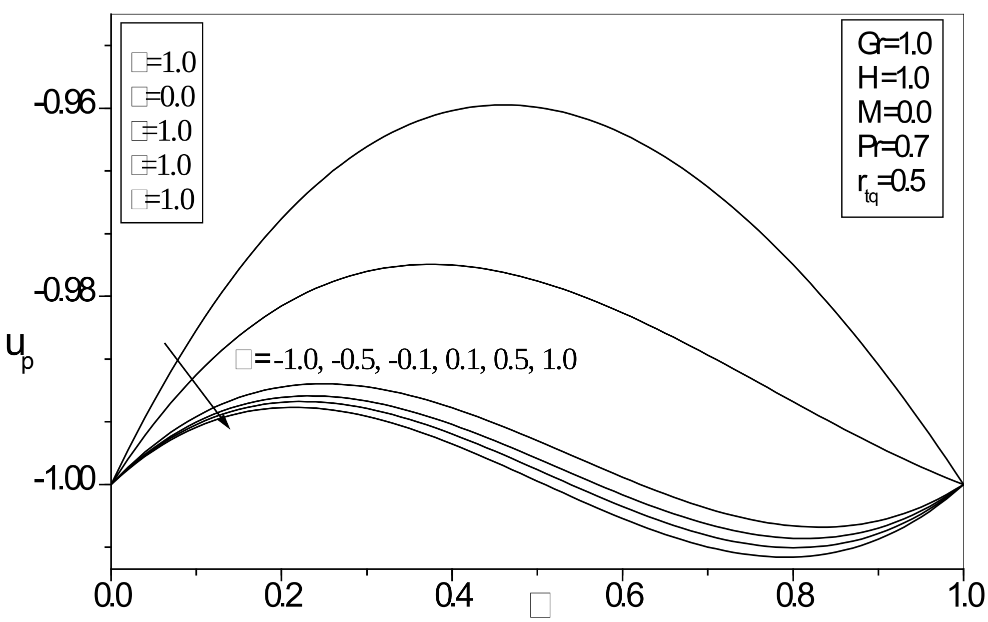

Figure 12. Effects of ϕ on particle phase velocity profile.

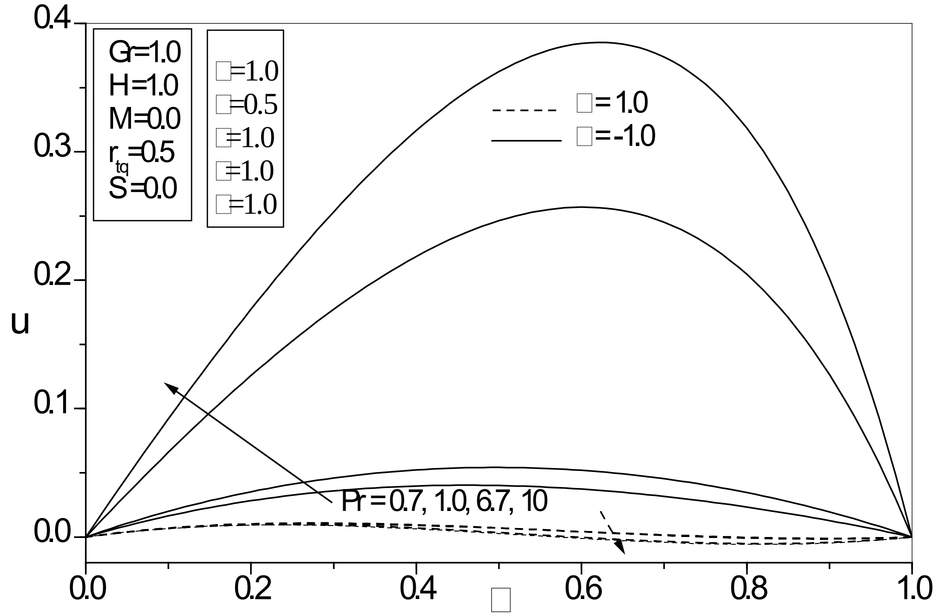

Figure 13. Effects of Pr on fluid phase velocity profile.

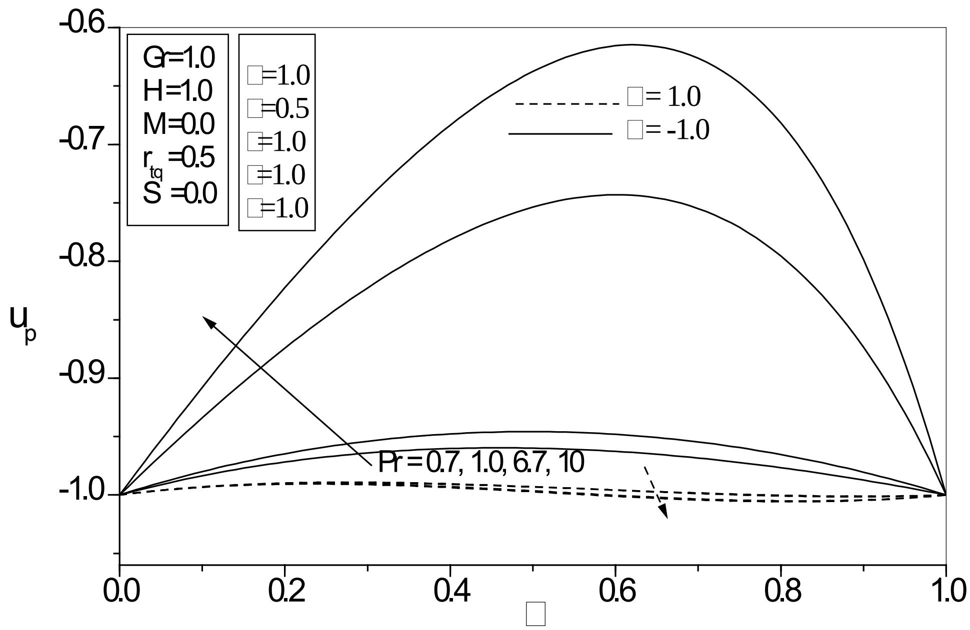

Figure 14. Effects of Pr on particle phase velocity profile.

Case 5: This case considers steady natural convection two-phase flow through an isothemal-isoflux channel in the presence of a particle-phase viscosity without magnetic field (M=0), and without heat generation or absorption ($\phi = 0$). The governing equation for this case will be the same as those of Case 1 except for the particle-phase momentum equation and its boundary conditions which can be written as:

$$\beta D^2 u_p + \alpha(u - u_p) - H = 0 \tag{52}$$

$$u_p(0) = S\, Du_p(0) - \frac{H}{\alpha} \tag{53a}$$

$$u_p(1) = -S\, Du_p(1) - \frac{H}{\alpha} \tag{53b}$$

where D is a first-order derivative operator with respect to η .

Now, we can start solving this problem by substituting equation (30) into equation (22) to obtain

$$D^2 u - \alpha\kappa\left(u - u_p\right) + Gr\left(-\eta + r_{tq}\right) + \kappa H = 0 \tag{54}$$

In operational form, equations (54) and (52)can be rearranged and rewritten respectively as

$$(D^2 - \alpha\kappa)\, u + \alpha\kappa\; u_p = -Gr\,(-\eta + r_{tq}) - \kappa H \tag{55}$$

$$\left(\frac{\alpha}{\beta}\right) u + (D^2 - \frac{\alpha}{\beta})\, u_p = \frac{H}{\beta} \tag{56}$$

In matrix form, these equations can be represented as

$$\begin{bmatrix} D^2 - \alpha\kappa & \alpha\kappa \\ \dfrac{\alpha}{\beta} & D^2 - \dfrac{\alpha}{\beta} \end{bmatrix} \begin{bmatrix} u \\ u_p \end{bmatrix} = \begin{bmatrix} -Gr(-\eta + r_{tq}) - \kappa H \\ \dfrac{H}{\beta} \end{bmatrix} \tag{57}$$

Let Δ be the following operator:

$$\Delta = \begin{vmatrix} D^2 - \alpha\kappa & \alpha\kappa \\ \dfrac{\alpha}{\beta} & D^2 - \dfrac{\alpha}{\beta} \end{vmatrix} = D^4 - (\frac{\alpha}{\beta} + \alpha\kappa)D^2 \tag{58}$$

Equations (55) and (56) can be written formally in the determinantal form respectively as

$$\Delta u = \begin{vmatrix} -Gr(-\eta + r_{tq}) - \kappa H & \alpha\kappa \\ \dfrac{H}{\beta} & D^2 - \dfrac{\alpha}{\beta} \end{vmatrix} \tag{59}$$

$$\Delta u_p = \begin{vmatrix} D^2 - \alpha\kappa & -Gr(-\eta + r_{tq}) - \kappa H \\ \dfrac{\alpha}{\beta} & \dfrac{H}{\beta} \end{vmatrix} \tag{60}$$

Equations (59) and (60) imply that

$$\left[D^4 - (\frac{\alpha}{\beta} + \alpha\kappa) D^2 \right] \text{u} = \frac{\alpha}{\beta} \text{ Gr } (-\eta + r_{tq}) \tag{61}$$

$$\left[D^4 - (\frac{\alpha}{\beta} + \alpha\kappa) D^2 \right] \text{u}_\text{p} = \frac{\alpha}{\beta} \text{ Gr(- } \eta + r_{tq}) \tag{62}$$

The general solutions of the above equations are

$$\text{u}(\eta) = c_1 + c_2\eta + c_3 e^{\xi\eta} + c_4 e^{\xi\eta} + Gr\left[\frac{\eta^3}{6} - \frac{\eta}{2} r_{tq} \right] / (1 + \beta\kappa) \tag{63}$$

$$\text{u}_\text{p}(\eta) = d_1 + d_2\eta + d_3 e^{\xi\eta} + d_4 e^{-\xi\eta} + Gr\left[\frac{\eta^3}{6} - \frac{\eta^2}{2} r_{tq} \right] / \text{ (1+ } \beta\kappa) \tag{64}$$

where

$$\zeta = (\frac{\alpha}{\beta} + \alpha\kappa)^{1/2} \tag{65}$$

The total number of independent constants present in the solution of the set of linear differential equations (61) and (62) is equal to the order of the operator Δ. Therefore, to determine the relationships which must exit among the c's and d's, we substitute equations (63) and (64) into equations (65) or equation (56), to obtain

$$\text{c}_1 = \text{d}_1 + \frac{H}{\alpha} - Gr \quad r_{tq} (\frac{\beta}{\alpha + \alpha\kappa\beta}) \tag{66}$$

$$c_2 = d_2 + Gr\left(\frac{-\beta}{\alpha + \alpha\kappa\beta}\right) \quad (67)$$

$$c_3 = \left(\frac{\alpha\kappa}{\alpha\kappa - \zeta^2}\right) d_3 \quad (68)$$

$$c_4 = \left(\frac{\alpha\kappa}{\alpha\kappa - \zeta^2}\right) d_4 \quad (69)$$

Now, by substituting these relations into equation (61) with the assumptions that

$$\psi = \frac{\alpha\kappa}{\alpha\kappa - \zeta^2} \quad (70)$$

gives the following general solution

$$u(\eta) = d_1 + \frac{H}{\alpha} - Gr\ r_{tq}\left(\frac{\beta}{\alpha + \alpha\beta\kappa}\right) + \left[d_2 + Gr\left(\frac{-\beta}{\alpha + \alpha\kappa\beta}\right)\right] + \psi d_3 e^{\xi\eta} + \psi d_4 e^{-\xi\eta}$$

$$+ Gr\left[\frac{\eta^3}{6} - \frac{\eta^2}{2} r_{tq}\right]/(1 + \beta\kappa) \quad (71)$$

In order to determine the constants d_1 through d_4, the boundary conditions (26a,b) and (53a,b) must be applied with equations (71) and (64) to give the following set of equations:

$$d_1 + \psi d_3 + \psi d_4 = Gr\ r_{tq}\left(\frac{\beta}{\alpha + \alpha\kappa\beta}\right) - \frac{H}{\alpha} \quad (72)$$

$$d_1 + d_2 + \psi d_3 e^{\zeta} + \psi d_4 e^{-\xi} = Gr\left[r_{tq}\left(\frac{1}{2} + \frac{\beta}{\alpha}\right) + \frac{\beta}{\alpha} - \frac{1}{6}\right]/(1 + \beta\kappa) - \frac{H}{\alpha} \quad (73)$$

$$d_1 - Sd_2 + (1 - S\zeta)d_3 + (1 + S\zeta)d_4 = -\frac{H}{\alpha} \quad (74)$$

$$d_1 + (1 + S)d_2 + (1 + S\zeta)\ d_3 e^{\xi} + (1 - S\zeta)\ d_4 e^{-\xi} =$$

$$-Gr\left(\frac{1 - 3r_{tq}}{6 + 6\beta\kappa}\right) - S\,Gr\left(\frac{1 - 2r_{tq}}{2 + 2\beta\kappa}\right) - \frac{H}{\alpha} \quad (75)$$

The above equations (72) through (75) determine d_1, d_2, d_3 and d_4. This concludes the solution and shows the effect of the slip coefficient S on the velocity profiles of both fluid and particle phases.

Within the range $0 \leq S \leq \infty$. Moreover, if $S \rightarrow \infty$, then equations (74) and (75) will be replaced by

$$d_2 + \zeta d_3 - \zeta d_4 = 0 \tag{76}$$

$$d_2 + \zeta e^{\zeta} d_3 - \zeta e^{-\zeta} d_4 = -\mathrm{Gr}\left(\frac{1-2r_{tq}}{2+2\beta\kappa}\right) \tag{77}$$

Figure 15 shows that increasing the particle-phase slip coefficient S has a significant increasing effect on the fluid-phase velocity profile. Moreover, as the particle-phase slip coefficient S increases, it becomes easier for the fluid to move causing the particle-phase velocity to increase as illustrated by Figure 16.

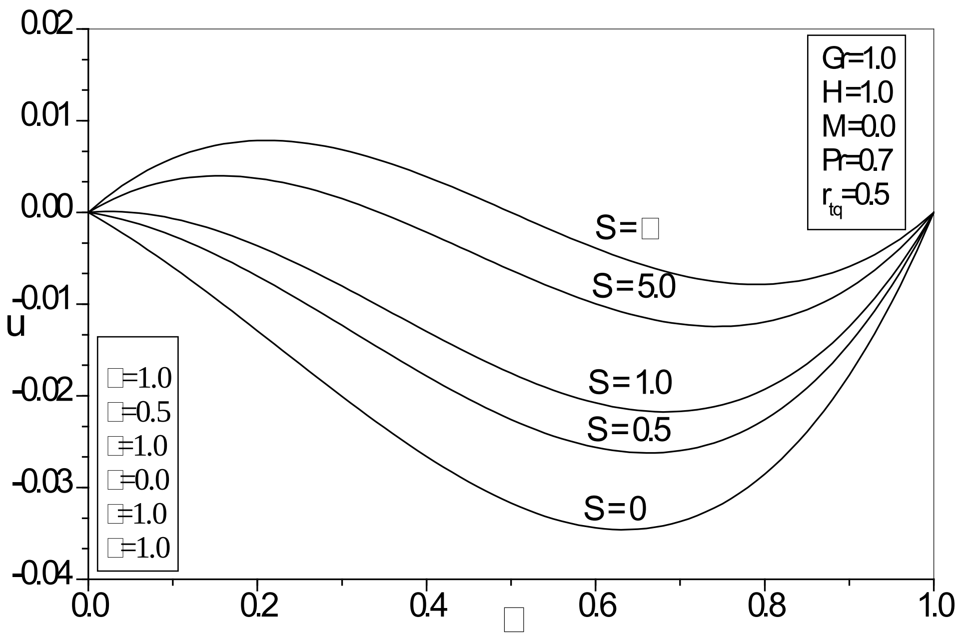

Figure 15. Effects of S on fluid phase velocity profile.

Case 6: This case considers steady natural convection two-phase flow through an isothermal-isoflux vertical channel in the presence of a heat generation source ($\phi > 0$), a magnetic field and a particle-phase viscosity.

The governing equations for this general steady-state problem in the absence of viscous dissipation and drag work can be written as

$$D^2u - \alpha\kappa(u-u_p) + Gr\theta - M^2u + \kappa \mathrm{H} = 0 \tag{78}$$

$$\frac{1}{\Pr} D^2\theta + \kappa\gamma\varepsilon\ (\theta_p - \theta) - \phi\theta = 0 \tag{79}$$

$$\beta D^2 u_p + \alpha(u - u_p) - H = 0 \tag{80}$$

$$\varepsilon(\theta_p - \theta) = 0 \tag{81}$$

The boundary conditions for this case can be written as follows

$$u(0) = u(1) = 0 \tag{82a,b}$$

$$\theta(1) = r_{tq} \tag{82c}$$

$$D\theta(1) = -1 \tag{82d}$$

$$u_p(0) = S\ Du_p(0) - \frac{H}{\alpha} \tag{82e}$$

$$u_p(1) = S\ Du_p(1) - \frac{H}{\alpha} \tag{82f}$$

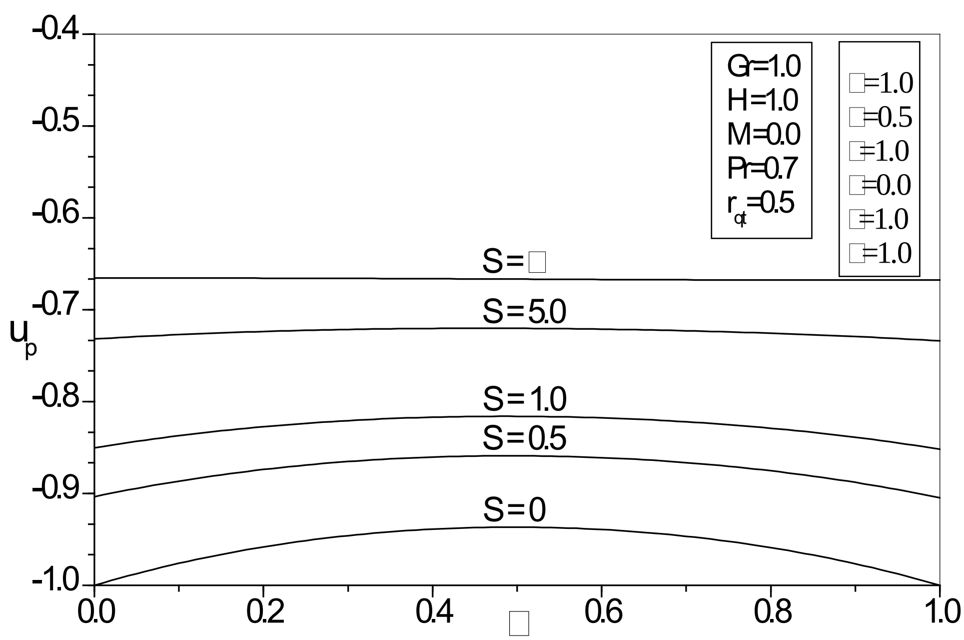

Figure 16. Effects of S on particle phase velocity profile.

Combining equations (81) and (79) and then solving for the fluid-phase temperature θ subject to the corresponding boundary conditions yields the following fluid-phase temperature profile:

$$\theta(\eta) = \left(\frac{-1 - r_{tq}\sqrt{\phi \Pr}\sinh\sqrt{\phi \Pr}}{\sqrt{\phi \Pr}\cosh\sqrt{\phi \Pr}} \right) \sinh\sqrt{\phi \Pr}\,\eta + r_{tq}\cosh\sqrt{\phi \Pr}\eta \tag{83}$$

According to equation (81), the particle-phase temperature profile will be the same as that of the fluid-phase. Equations (78) and (80) can be rearranged and rewritten in matrix form as follows:

$$\begin{bmatrix} D^2 - \alpha\kappa - M^2 & \alpha\kappa \\ \dfrac{\alpha}{\beta} & D^2 - \dfrac{\alpha}{\beta} \end{bmatrix} \begin{bmatrix} u \\ u_p \end{bmatrix} = \begin{bmatrix} -Gr\theta(\eta) - \kappa H \\ \dfrac{H}{\beta} \end{bmatrix} \tag{84}$$

where $\theta(\eta)$ is given by equation (83) .

Let Δ be the following operator:

$$\Delta = \begin{vmatrix} D^2 - \alpha\kappa - M^2 & \alpha\kappa \\ \dfrac{\alpha}{\beta} & D^2 - \dfrac{\alpha}{\beta} \end{vmatrix} = D^4 - \left(\frac{\alpha}{\beta} + \alpha\kappa + M^2\right) D^2 + \frac{\alpha}{\beta}M^2 \tag{85}$$

The matrix form represents a system of two second-order differential equations with two unknowns. These equations can be rewritten with the aid of determinants as follows:

$$\Delta u = \begin{vmatrix} -Gr\theta - \kappa H & \alpha\kappa \\ \dfrac{H}{\beta} & D^2 - \dfrac{\alpha}{\beta} \end{vmatrix} \tag{86}$$

$$\Delta u_p = \begin{vmatrix} D^2 - \alpha\kappa - M^2 & -Gr\theta - \kappa H \\ \dfrac{\alpha}{\beta} & \dfrac{H}{\beta} \end{vmatrix} \tag{87}$$

Equation (86) and (87) can be reduced to the following forms:

$$\left[D^2 - \left(\frac{\alpha}{\beta} + \alpha\kappa + M^2\right) D^2 + \frac{\alpha}{\beta}M^2 \right] \mathrm{u} = \left(D^2 - \frac{\alpha}{\beta} \right)\left(-Gr\theta - \kappa H \right) - \alpha\kappa\frac{H}{\beta} \tag{88}$$

$$\left[D^2-(\frac{\alpha}{\beta}+\alpha\kappa+M^2)D^2+\frac{\alpha}{\beta}M^2\right]u_p = \frac{\alpha}{\beta}\ \mathrm{Gr}\ \theta(\eta)-\frac{H}{\beta}M^2 \tag{89}$$

where $\theta(\eta)$ is given by equation (83) .

The above equations can be solved by the usual method of solving such equations to give the following general solutions:

$$u(\eta) = c_1e^{\zeta_1\eta} + c_2e^{-\zeta_1\eta} + c_3e^{\zeta_2\eta} + c_4e^{-\zeta_2\eta} + X_1\ \theta(\eta) \tag{90}$$

$$u_p(\eta) = d_1e^{\xi_1\eta} + d_2e^{-\xi_1\eta} + d_3e^{\xi_2\eta} + d_4e^{-\xi_2\eta} + X_2\ \theta(\eta) - \frac{H}{\alpha} \tag{91}$$

where

$$\zeta_1 = \sqrt{\frac{\frac{\alpha}{\beta}+\alpha\kappa+M^2+\sqrt{\left(\frac{\alpha}{\beta}+\alpha\kappa+M^2\right)^2-4\alpha\frac{M^2}{\beta}}}{2}} \tag{92}$$

$$\zeta_2 = \sqrt{\frac{\frac{\alpha}{\beta}+\alpha\kappa+M^2-\sqrt{\left(\frac{\alpha}{\beta}+\alpha\kappa+M^2\right)^2-4\alpha\frac{M^2}{\beta}}}{2}} \tag{93}$$

$$X_1 = \frac{\left(\frac{\alpha}{\beta}-\phi\,\mathrm{Pr}\right)Gr}{(\phi\,\mathrm{Pr})^2-(\frac{\alpha}{\beta}+\alpha\kappa+M^2)\phi\,\mathrm{Pr}+\alpha\frac{M^2}{\beta}} \tag{94}$$

$$X_2 = \frac{\frac{\alpha}{\beta}Gr}{(\phi\,\mathrm{Pr})^2-(\frac{\alpha}{\beta}+\alpha\kappa+M^2)\phi\,\mathrm{Pr}+\alpha\frac{M^2}{\beta}} \tag{95}$$

As in Case 5 again the relations which must exist among the constants c’s and d’s can be represented as:

$$c_1 = \left(1-\frac{\beta}{\alpha}\zeta_1^2\right)d_1 \tag{96}$$

$$c_2 = \left(1 - \frac{\beta}{\alpha}\zeta_1^2\right)d_2 \tag{97}$$

$$c_3 = \left(1 - \frac{\beta}{\alpha}\zeta_2^2\right)d_3 \tag{98}$$

$$c_4 = \left(1 - \frac{\beta}{\alpha}\zeta_2^2\right)d_4 \tag{99}$$

Substituting the above relations back into equation (90) with the assumption that

$$\psi_1 = \left(1 - \frac{\beta}{\alpha}\zeta_1^2\right) \tag{100}$$

$$\psi_2 = \left(1 - \frac{\beta}{\alpha}\zeta_2^2\right) \tag{101}$$

gives the following general solutions for the fluid-phase velocity:

$$u(\eta) = \psi_1 d_1 e^{\zeta_1\eta} + \psi_1 d_2 e^{-\zeta_1\eta} + \psi_2 d_3 e^{\zeta_2\eta} + \psi_2 d_4 e^{-\zeta_2\eta} + X_1\ \theta(\eta) \tag{102}$$

The values of the constants d's can be determined by applying the velocities boundary conditions (82a,b,e,f) with their corresponding equations (102) and (91) to give

$$\psi_1 d_1 + \psi_1 d_2 + \psi_2 d_3 + \psi_2 d_4 + X_1\ r_{tq} = 0 \tag{103}$$

$$\psi_1 d_1 e^{\zeta_1} + \psi_1 d_2 e^{-\zeta_1} + \psi_2 d_3 e^{\zeta_2} + \psi_2 d_4 e^{-\zeta_2} + X_1\ r_{tq} = 0 \tag{104}$$

$$(1-S\zeta_1)d_1 + (1+S\zeta_1)d_2 + (1-S\zeta_2)d_3 + (1+S\zeta_2)d_4 = -\left[r_{tq} + S\left(\frac{1 + r_{tq}\sqrt{\phi\,\mathrm{Pr}}\sinh\sqrt{\phi\,\mathrm{Pr}}}{\cosh\sqrt{\phi\,\mathrm{Pr}}}\right)\right] \tag{105}$$

$$(1+S\zeta_1)d_1 e^{\zeta_1} + (1-S\zeta_1)d_2 e^{-\zeta_1} + (1+S\zeta_2)d_3 e^{\zeta_2} + (1-S\zeta_2)d_4 e^{-\zeta_2} = X_2(S-\theta) \tag{106}$$

The above four equations can be solved simultaneously to determine the constants d_1, d_2, d_3, and d_4. With these constants known, then the fluid-phase velocity profile can be calculated from equation (102) . It should be reminded that the solution for the particle-phase velocity profile can be obtained from equation (91) , this solution is valid for $0 \le S \le \infty$. For the case of perfect particle-phase wall slip ($S \to \infty$), then equations (105) and (106) will be replaced by

$$\zeta_1 d_1 - \zeta_1 d_2 + \zeta_2 d_3 - \zeta_2 d_4 = X_2 \left[\frac{1 + r_{tq}\sqrt{\phi \Pr}\sinh\sqrt{\phi \Pr}}{\cosh\sqrt{\phi \Pr}} \right] \tag{107}$$

$$\zeta_1 e^{\zeta_1} d_1 - \zeta_1 e^{-\zeta_1} d_2 + \zeta_2 e^{\zeta_2} d_3 - \zeta_2 e^{-\zeta_2} d_4 = X_2 \tag{108}$$

<u>Case 7</u>: This case considers the same assumptions of Case 6 except in the presence of a heat absorbing sink ($\phi < 0$) instead of a heat generation source. The governing equations and the corresponding boundary conditions are the same as those in Case 6 except the fluid-phase energy equation, which can be written as:

$$\frac{1}{\Pr} D^2\theta + \kappa\gamma\varepsilon(\theta_p - \theta) + \phi\theta = 0 \tag{109}$$

Combining equations (109) and (81) yields

$$D^2\theta + \phi \Pr \ \theta = 0 \tag{110}$$

This equation can be solved subject to the boundary conditions given in equations (82 c,d) by the usual method of solving such equations to give the following fluid-phase temperature profile :

$$\theta(\eta) = r_{tq} \cos\sqrt{\phi \Pr}\ \eta - \left[\frac{1 + r_{tq}\sin\sqrt{\phi \Pr}}{\sqrt{\phi \Pr}\cos\sqrt{\phi \Pr}} \right] \sin\sqrt{\phi \Pr}\eta \tag{111}$$

The particle-phase temperature can be obtained from equation (81) to give

$$\theta_p(\eta) = r_{tq} \cos\sqrt{\phi \Pr}\ \eta - \left[\frac{1 + r_{tq}\sin\sqrt{\phi \Pr}}{\sqrt{\phi \Pr}\cos\sqrt{\phi \Pr}} \right] \sin\sqrt{\phi \Pr}\eta \tag{112}$$

Following the same procedure as that of Case 6 in finding the solution of the velocities of each phase, one can obtain the following general solutions for the fluid and particle-phase velocities, respectively as:

$$u(\eta) = c_1\ e^{\xi_1\eta} + c_2\ e^{-\xi_1\eta} + c_3\ e^{\xi_2\eta} + c_4\ e^{-\xi_2\eta} + X_1\theta(\eta) \tag{113}$$

$$u_p(\eta) = \mathrm{d}_1\; e^{\xi_1\eta} + \mathrm{d}_2\; e^{-\xi_1\eta} + \mathrm{d}_3\; e^{\xi_2\eta} + \mathrm{d}_4\; e^{-\xi_2\eta} + \mathrm{X}_2\;\theta(\eta) - \frac{H}{\alpha} \tag{114}$$

where

$$\zeta_1 = \sqrt{\frac{\frac{\alpha}{\beta} + \alpha\kappa + M^2 + \sqrt{\left(\frac{\alpha}{\beta} + \alpha\kappa + M^2\right)^2 - 4\alpha\frac{M^2}{\beta}}}{2}} \tag{115}$$

$$\zeta_2 = \sqrt{\frac{\frac{\alpha}{\beta} + \alpha\kappa + M^2 - \sqrt{\left(\frac{\alpha}{\beta} + \alpha\kappa + M^2\right)^2 - 4\alpha\frac{M^2}{\beta}}}{2}} \tag{116}$$

$$\mathrm{X}_1 = \frac{\left(\frac{\alpha}{\beta} + \phi\,\mathrm{Pr}\right)Gr}{(\phi\,\mathrm{Pr})^2 + (\frac{\alpha}{\beta} + \alpha\kappa + M^2)\phi\,\mathrm{Pr} + \alpha\frac{M^2}{\beta}} \tag{117}$$

$$\mathrm{X}_2 = \frac{\frac{\alpha}{\beta}Gr}{(\phi\,\mathrm{Pr})^2 + (\frac{\alpha}{\beta} + \alpha\kappa + M^2)\phi\,\mathrm{Pr} + \alpha\frac{M^2}{\beta}} \tag{118}$$

where $\theta(\eta)$ is given by equation (111) .

The relationships between the c's and d's are the same as those in Case 6 which are equations (96-99). Substituting these relations back into equation (113) with assuming that

$$\psi_1 = \left(1 - \frac{\beta}{\alpha}\zeta_1^2\right) \tag{119}$$

$$\psi_2 = \left(1 - \frac{\beta}{\alpha}\zeta_2^2\right) \tag{120}$$

implies that the general solution of the fluid-phase velocity will be given by

$$u(\eta) = \psi_1 d_1 e^{\xi_1\eta} + \psi_1 d_2 e^{-\xi_1\eta} + \psi_2 d_3 e^{\xi_2\eta} + \psi_2 d_4 e^{-\xi_2\eta} + \mathrm{X}_1\;\theta(\eta) \tag{121}$$

From equations (121) and (114) and their boundary conditions, (82a,b,e,f), a system of equations for the d's can be obtained. Again, as previously mentioned, the constants d's can be determined by solving this system of equations

$$\psi_1 d_1 + \psi_1 d_2 + \psi_2 d_3 + \psi_2 d_4 = -X_1 \, r_{tq} \tag{122}$$

$$\psi_1 d_1 e^{\xi_1} + \psi_1 d_2 e^{-\xi_1} + \psi_2 d_3 e^{\xi_2} + \psi_2 d_4 e^{-\xi_2} = -X_1 \, \theta(\eta) \tag{123}$$

$$(1 - S\zeta_1)d_1 + (1+S\zeta_1)d_2 + (1 - S\zeta_2)d_3 + (1+S\zeta_2)d_4 = -X_2\left[r_{tq} + S\left(\frac{1 + r_{tq}\sqrt{\phi \Pr}\sin\sqrt{\phi \Pr}}{\cos\sqrt{\phi \Pr}}\right)\right] \tag{124}$$

$$(1 + S\zeta_1)d_1 e^{\zeta_1} + (1- S\zeta_1)d_2 e^{-\zeta_1} + (1 + S\zeta_2)d_3 e^{\zeta_2} + (1- S\zeta_2)d_4 e^{-\zeta_2} = X_2 S - X_2\,\theta(\eta) \tag{125}$$

This system of equations is valid within the range $0 \le S \le \infty$. Moreover, in the limit $S \to \infty$, equations (124) and (125) will be replaced by

$$\zeta_1 d_1 - \zeta_1 d_2 + \zeta_2 d_3 - \zeta_2 d_4 = X_2\left[\frac{1 + r_{tq}\sqrt{\phi \Pr}\sin\sqrt{\phi \Pr}}{\cos\sqrt{\phi \Pr}}\right] \tag{126}$$

$$\zeta_1 e^{\zeta_1} d_1 - \zeta_1 e^{-\zeta_1} d_2 + \zeta_2 e^{\zeta_2} d_3 - \zeta_2 e^{-\zeta_2} d_4 = X_2 \tag{127}$$

In order to elucidate the influence of the inverse Stokes number α, graphical representation of u and u_p is obtained and presented in Figures 17 and 18. As α increases , both the fluid-phase and the particle-phase velocities increase.

Figures 19 and 20 show representative velocity profiles for the fluid and particle phases for various values of the particle loading κ, respectively. Physically speaking, as the particle loading in the channel increases, the drag force between the phases increases causing a slower motion of the fluid.

This produce a reduction in the particle-phase velocity since the particle phase is being dragged along by the fluid. It should be noted that for $\kappa = 100$ the fluid-phase velocity is small as it becomes harder for the flow to move through the channel.

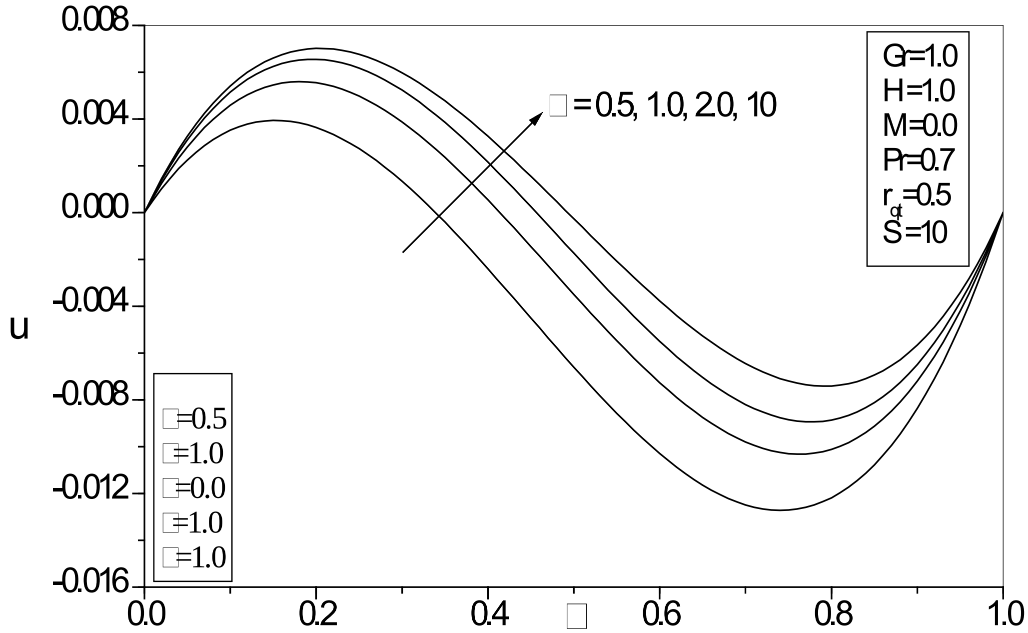

Figure 17. Effects of α on fluid phase velocity profile.

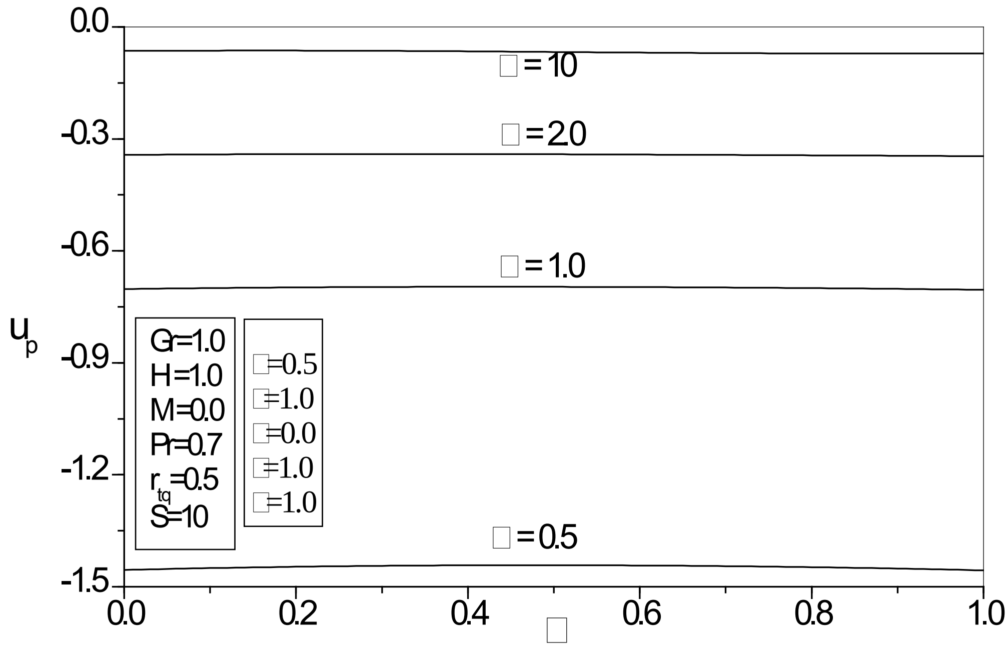

Figure 18. Effects of α on particle phase velocity profile.

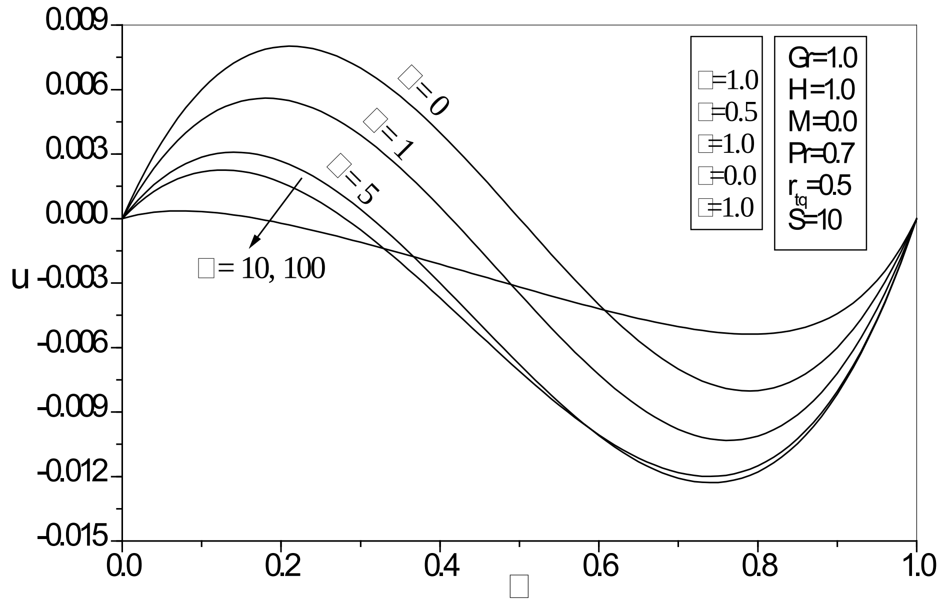

Figure 19. Effects of κ on fluid phase velocity profile.

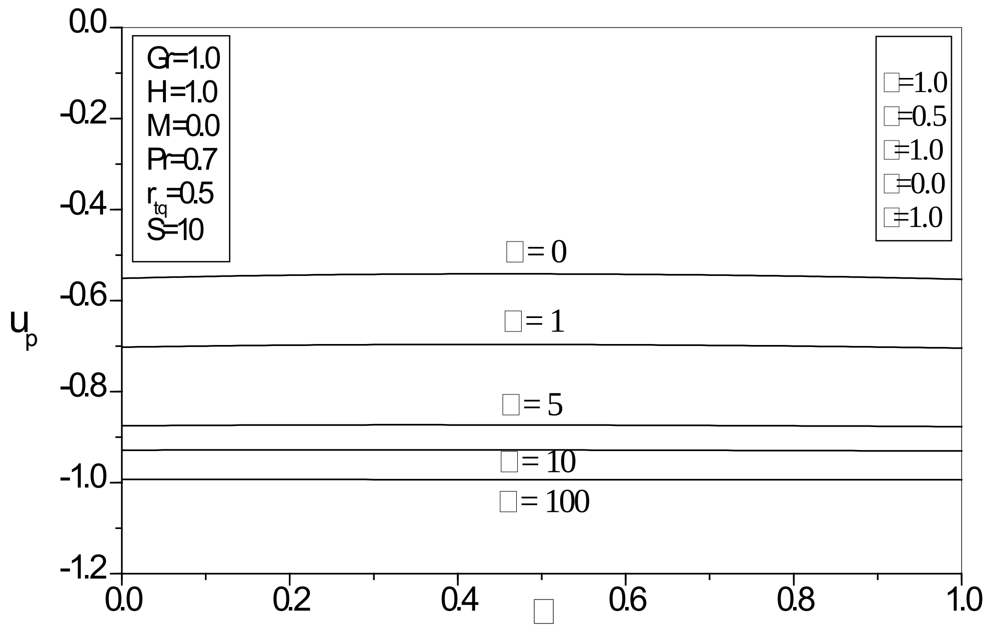

Figure 20. Effects of κ on particle phase velocity profile.

CONCLUSION

A continuum model for natural convection flow of a two-phase particulate suspension was formulated and applied to the problem of steady fully-developed natural convection flow through a vertical parallel-plate channel heated asymmetrically. The general formulation took into account the effects of particle-phase viscosity, magnetic field, and heat generation or absorption effects. The governing equations were non-dimensionalized and solved analytically for the isoflux-isothermal boundary conditions of the channel walls. Closed-form solutions for some special and general cases were reported and representative results were plotted to illustrate the influence of the physical parameters on the solutions. Based on the results, the following conclusions can be summarized:

1. In the absence of viscous and magnetic dissipation, drag work, and heat generation or absorption, the temperature profiles of both phases were identical and had a linear shape of pure conduction.
2. Increases in the particle concentration (particle loading) increased the drag force between the phases causing a slower motion of both the fluid and particle phases.
3. Increasing in the values of the Grashof number which represents the driving force led to increases in the flow of both phases.
4. The magnetic field had the effect of reducing the velocity of the fluid-phase which, in turn, reduced the particle-phase velocity.
5. The particle-phase wall slip coefficient had a limited increasing effect on the fluid-phase velocity, while the particle-phase velocity increased significantly.
6. Increases in the values of the heat generation (or absorption) coefficient caused the fluid temperature to increase (decrease).
7. An increase in the values of wall thermal ratio increased the fluid temperature and therefore, the thermal buoyancy effect which, consequently, increased the flow of both phases in the channel.

It is hoped that the analytical results obtained in this work be used as a vehicle for understanding more complex situations of natural convection in two-phase suspensions and a stimulus for experimental work.

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