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Existence of multiple solutions for magnetohydrodynamic flows of second-grade and Walter's B fluids due continuously contracting flat sheet with partial slip

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ABSTRACT

In the present investigation, an attempt is performed to analyse the existence of multiple exact solutions for magnetohydrodynamic flows of second-grade and Walter's B fluids due continuously contracting flat sheet with partial slip condition at the boundary. After transforming basic governing equations by suitable similarity transformations, single and double closed-form exact solutions of the flow are achieved on certain restrictions on flow variables of two aforesaid fluids. The impacts of magnetic field and partial slip on existence and non-existence of solutions are revealed. The changes of flow surface drag and velocity with various involved parametric values are also explored. For both viscoelastic type fluids, i.e., second-grade and Walter's B fluids, double solutions exist for certain flow conditions. In second-grade case, if viscoelastic parameter exceeds a definite value, then double similarity solutions are achieved for any type of flow, i.e., slip flow or no-slip flow. For Walter's B fluid similar results are obtained with unique similarity slip flow solution if magnitude of viscoelastic parameter exceeds certain fixed value. Importantly, the magnetic field and its MHD effect stabilize the uncertainty situation in existence of similarity solution for both fluids and produce unique solution for a suitable choice.

1. Introduction

The fields of the applications of fluid flow over expanding/ contracting sheet were stated by Karwe and Jaluria[1] and Sparrow and Abraham.[2] Examples of engineering and industrial applications of expanding/contracting sheet are drawing of annealing and tinning of copper wires, continuous filaments through quiescent fluids, wire drawing and glass–fibre production, manufacturing of plastic and rubber sheets, glass blowing, manufacturing and extraction of polymer and rubber sheets and crystal growing, etc. The closed form analytical solution of flow over expanding sheet was obtained by Crane.[3] The heat and mass transfer aspects of Crane's problem were analysed by Gupta and Gupta.[4] The self-similar exact solution for the flow over permeable expanding sheet was obtained by Magyari and Keller.[5] Wang[6] obtained exact similarity solution for flow over expanding sheet in 3-Dimension. Whereas, Wang[7] also obtained condition that mass suction is needed for existence of solution for flow over contracting sheet and this fact is completely innovative in context of outcome of expanding sheet flow. Later, detailed existence and non-uniqueness of solutions for Newtonian flow on contracting sheet was introduced by Miklavčič and Wang.[8] Fang and Zhang[9] acquired analytical solution in closed form for thermal boundary layer over contracting sheet. After these investigations many researchers (Khan et al.,[10] Fang et al.,[11,12] Bhattacharyya,[13] Bhattacharyya et al.,[14] Bhatti et al.,[15] Jumana et al.,[16] Tshivhi et al.,[17] Dey et al.,[18] and Rajput et al.[19,20]) obtained multiple solutions in their studies. Boundary layer flow(BLF) over contracting sheet is quite different than expanding sheet and it shows distinct physical behaviours. For steady flow due to contraction of flat sheet, the generated vorticity is not limited inside boundary layer unless suitable suction is adopted. The flow over contracting sheet may be utilized on rising contracting balloon, and contracting film, etc.

All the above studies considered Newtonian fluid model, whereas the study of dynamics of non-Newtonian fluid flow is vital due to its widespread utilizations in food processing, plastic manufacturing process, geophysical and biological fluids and performance of lubricants and paints. The non-Newtonian fluid models having viscous and elastic properties are familiar as viscoelastic fluids. Rajagopal et al.[21] and Dandapat and Gupta[22] considered viscoelastic fluid flow models over

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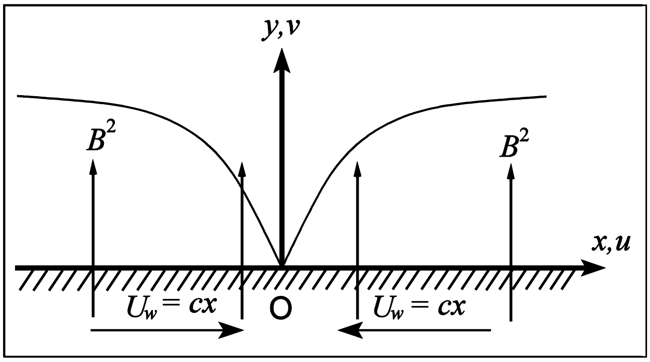


Fig. 1. Physical sketch of the mathematical model.

expanding sheet. The analytical solution of viscoelastic fluid flow over exponential expanding sheet was reported by Khan and Sanjayanand.[23] Khan[24] obtained closed from analytical solution of viscoelastic fluid with suction and thermal radiation over expanding sheet. The flow of second-grade fluid with temperature dependent viscosity was analysed by Nadeem and Faraz.[25] Flow of second-grade fluid over contracting sheet was examined by Nadeem et al.[26] The stability analysis for impinging oblique flow of viscoelastic fluid towards a contracting sheet near a stagnation-point was described by Naganthran et al.[27] Analytical closed form solution of second-grade fluid over expanding sheet was obtained by Ghadikolaei et al.[28] and Rafiq et al.[29] Recently, many researchers (Bataller,[30] Cortell,[31] Mustafa,[32] Hussain et al.,[33] Galeel et al.,[34] Li and Liu[35] and Damala et al.[36]) analysed several vital aspects of viscoelastic flows.

The theory of magnetohydrodynamics (MHD) has significant role in developing of BLF. The BLF of viscoelastic fluid over expanding sheet with MHD effect has massive importance in petroleum and chemical engineering processes. The applications of such type of flows are found in manufacturing of rubber sheets, polymer extrusion, drawing of plastic films, etc. Pavlov[37] first obtained exact similarity solution of MHD electrically conducting fluid flow over deformable plane surface. Problem of Crane[3] was extended by Chakrabarti and Gupta[38] with inclusion of MHD effect. Chen[39] demonstrated closed form solutions of MHD flow for two viscoelastic type fluids over expanding sheet. The closed form analytical solutions of several MHD flow over contracting/expanding sheet were reported by Hayat et al.,[40] Fang and Zhang,[41] whereas numerical solutions by Muhaimin et al.,[42] Bhattacharyya[43] and Aloliga and Azuure.[44] Some new results on MHD non-Newtonian flow induced by expanding/contracting sheet were described by Mahabaleshwar et al.,[45] Roy and Pop,[46] Talla et al.[47] and Dey and Borah.[48]

The effect of slip is neglected in all above research problems, however consideration of this phenomenon altering conventional no-slip case is always interesting in fluid mechanics. In many technological processes, like design of various microfluids systems and red blood flow through capillaries, slip BLF occurs. To get a complete understanding of aforesaid flow characteristics, the inherent property of interface velocity of fluid-solid needs to be addressed. Boundary slip phenomenon was proposed by Navier,[49] whereas Andersson[50] explained slip flow over expanding sheet. Analytical solution for slip boundary layer due to expansion of sheet was achieved by Wang.[51] Ariel et al.[52] and Hayat et al.[53] acquired exact analytical solutions for Walter's B liquid and second-grade fluid with slip, respectively and Sahoo[54] did it for third grade fluid along with partial slip. Fang et al.[55] reported exact analytical solution for MHD slip flow on expanding sheet and Hayat et al.[46] discussed heat transfer of this flow. Turkyilmazoglu[56,57] established conditions for existence of multiple solutions for two types of viscoelastic fluids on expanding sheet with MHD and partial slip. Fang et al.[58,59] discussed second-order slip flow over contracting sheet for Newtonian fluid case and they also incorporated MHD impact on aforesaid flow. Some more important features of slip flow on expanding/contracting sheet may be found in literature.[60–65]

Motivated by unusual behaviour of contracting flow and importance of viscoelastic fluids, a challenge of analysing 2D, viscous, incompressible, steady, laminar, MHD BLF of two types of viscoelastic fluids over linear contracting sheet is undertaken. To maintain steady flow on contracting porous sheet, mass suction usually assumed, whereas in present analysis magnetic field and partial slip are simultaneously considered to explore conditions of existence and non-existence of steady-state solution. Also, dependence of uniqueness and non-uniqueness of solutions on magnetic field and partial slip are examined. During the analysis, exact similarity solutions are obtained. The novelty of the investigation is inside the consideration of partial slip and magnetic field instead of fluid mass suction through porous sheet. So, in absence of mass suction how the slip phenomenon and external magnetic field work together in maintaining the steady-state boundary layer flow of two types of viscoelastic fluids by delaying the separation is explored. One important physical property of the sheet is also assumed, i.e., the sheet is non-porous. The authors are confident that the considered problem on two different viscoelastic fluids with above mentioned conditions will add some interesting information to the literature.

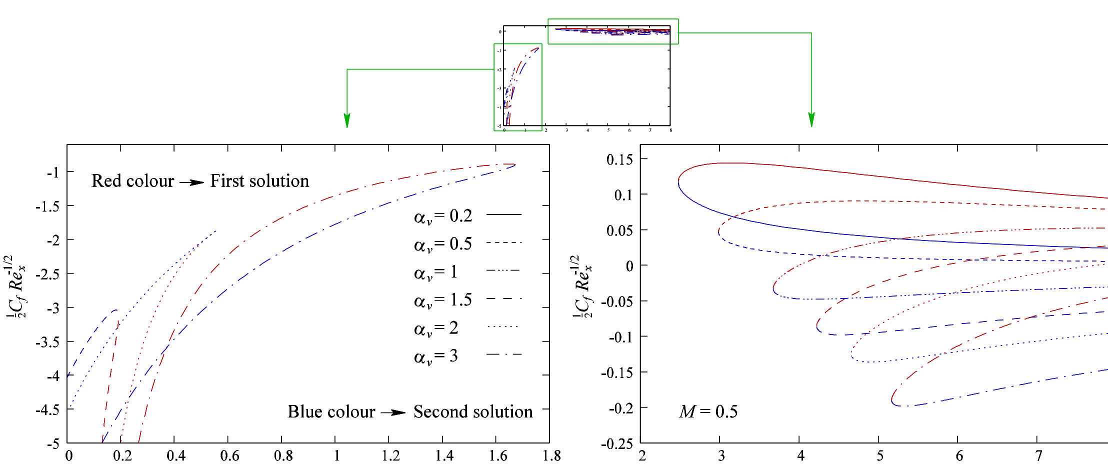


Fig. 2. $\frac{1}{2}Re_x^{-1/2}C_f$ vs. λ_1 for various α_v(second-grade fluid, $\alpha_v > 0$) with $M = 0.5$.

Table 1
The critical values and ranges of λ_1 for existence of double, single and no similarity solutions for second-grade fluid with various $\alpha_\upsilon(>0)$ when $M = 0.5$.

α_υ	Critical values of λ_1		Ranges of λ_1 with		
	λ_{1SG}	λ^*_{1SG}	Double solutions	Single solution	No solution
0	2.075911	–	[2.075911, ∞)	–	[0,2.075911)
0.2	2.477049	–	[2.477049, ∞)	–	[0,2.477049)
0.5	2.985366	–	[2.985366, ∞)	–	[0,2.985366)
1	3.674236	–	[3.674236, ∞)	–	[0,3.674236)
1.001	3.675460	0.000017	[3.675460, ∞) & [0,0.000017]	–	(0.000017,3.675460)
1.05	3.734799	0.006085	[3.734799, ∞) & [0,0.006085]	–	(0.006085,3.734799)
1.1	3.794036	0.017216	[3.794036, ∞) & [0,0.017216]	–	(0.017216,3.794036)
1.5	4.223462	0.193323	[4.223462, ∞) & [0,0.193323]	–	(0.193323,4.223462)
2	4.657540	0.554367	[4.657540, ∞) & [0,0.554367]	–	(0.554367,4.657540)
3	5.167583	1.672153	[5.167583, ∞) & [0,1.672153]	–	(1.672153,5.167583)
3.5	5.158702	2.502555	[5.158702, ∞) & [0,2.502555]	–	(2.502555,5.158702)
3.8	4.950767	3.204667	[4.950767, ∞) & [0,3.204667]	–	(3.204667,4.950767)
3.9	4.785103	3.535218	[4.785103, ∞) & [0,3.535218]	–	(3.535218,4.785103)
3.99	4.434151	4.034631	[4.434151, ∞) & [0,4.034631]	–	(4.034631,4.434151)
3.999	4.305032	4.178598	[4.305032, ∞) & [0,4.178598]	–	(4.178598,4.305032)
4	–	–	[0, ∞)		

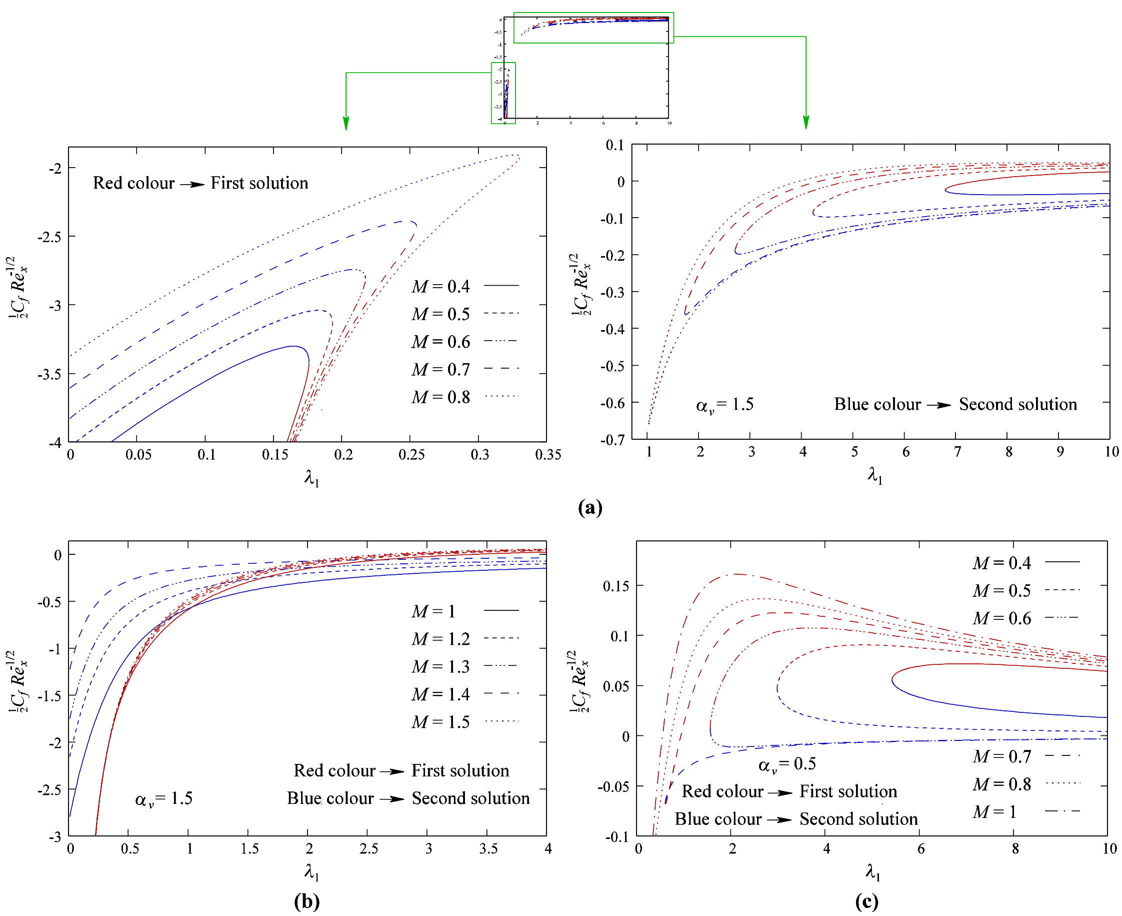


Fig. 3. $\frac{1}{2}Re_x^{-1/2}C_f$ vs. λ_1 for various M (second-grade fluid, $\alpha_\upsilon > 0$) with (a) $\alpha_\upsilon = 1.5$ (b) $\alpha_\upsilon = 1.5$ and (c) $\alpha_\upsilon = 0.5$.

2. Flow governing equations

Consider steady laminar motion of incompressible second-grade fluid and Walter's B fluid on a contracting sheet with imposed magnetic field and partial slip. The contracting flat sheet is taken along plane $y = 0$ and flow region is restricted in $y > 0$. Motion is induced for the contraction of flat sheet using application of two equal and reverse forces along x-axis protecting position of the origin. Boundary layer steady-state equations for motion are (Bataller,[30] Cortell[31])

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}\pm k^*\left[u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right]-\frac{\sigma B^2}{\rho}u, \tag{2.2}$$

where u and v are velocity components along x- and y-directions, respectively, υ is kinematic viscosity, "+" sign in RHS of Eq. (2.2) corresponds to second-grade fluid, whereas "-" sign stands for Walter's B fluid and $k^*(=\alpha_1^*/\rho;\ \alpha_1^*$ being non-negative material modulus and it is very small for Walter's B fluid) is coefficient of viscoelasticity,

Table 2
The critical values and ranges of λ_1 for existence of double, single and no similarity solutions for second-grade fluid with various M.

M	α_v	Critical values of λ_1		Ranges of λ_1 with		
		λ_{1SG}	λ^*_{1SG}	Double solutions	Single solution	No solution
0.3	0.5	10.238074	–	[10.238074,∞)	–	[0,10.238074)
	1.5	11.822233	0.162786	[11.822233,∞) & (0,0.162786]	{0}	(0.162786,11.822233)
0.4	0.5	5.422730	–	[5.422730,∞)	–	[0,5.422730)
	1.5	6.798456	0.176037	[6.798456,∞) & (0,0.176037]	{0}	(0.176037,6.798456)
0.5	0.5	2.985366	–	[2.985366,∞)	–	[0,2.985366)
	1.5	4.223462	0.193323	[4.223462,∞) & (0,0.193323]	{0}	(0.193323,4.223462)
0.6	0.5	1.563076	–	[1.563076,∞)	–	[0,1.563076)
	1.5	2.710827	0.217371	[2.710827,∞) & (0,0.217371]	{0}	(0.217371,2.710827)
0.7	0.5	0.608251	–	[0.608251,∞)	–	[0,0.608251)
	1.5	1.730462	0.254768	[1.730462,∞) & (0,0.254768]	{0}	(0.254768,1.730462)
0.74	0.5	0.228368	–	[0.228368,∞)	–	[0,0.228368)
0.749	0.5	0.069184	–	[0.069184,∞)	–	[0,0.069184)
0.7500	0.5	0.008605	–	[0.008605,∞)	–	[0,0.008605)
0.7501	0.5	–	–	–	(0,∞)	{0}
0.8	0.5	–	–	–	(0,∞)	{0}
	1.5	1.021334	0.330488	[1.021334,∞) & (0,0.330488]	{0}	(0.330488,1.021334)
0.85	1.5	0.663938	0.447311	[0.663938,∞) & (0,0.447311]	{0}	(0.447311,0.663938)
0.857	1.5	0.555436	0.525303	[0.555436,∞) & (0,0.525303]	{0}	(0.525303,0.555436)
0.8571	1.5	0.548403	0.531905	[0.548403,∞) & (0,0.531905]	{0}	(0.531905,0.548403)
0.8572	1.5	–	–	(0,∞)	{0}	–
0.858	1.5	–	–	(0,∞)	{0}	–
0.9	1.5	–	–	(0,∞)	{0}	–
1	0.5	–	–	–	(0,∞)	{0}
	1.5	–	–	(0,∞)	–	{0}
1.2	0.5	–	–	–	[0,∞)	–
	1.5	–	–	(0,∞)	–	{0}
1.4	1.5	–	–	(0,∞)	–	{0}
1.49	1.5	–	–	(0,∞)	–	{0}
1.4999	1.5	–	–	(0,∞)	–	{0}
1.5	1.5	–	–	–	(0,∞)	{0}
2	1.5	–	–	–	(0,∞)	{0}

Table 3
The critical values and ranges of λ_1 for existence of double, single and no similarity solutions for Walter's B fluid with various $\alpha_v(<0)$ when $M = 0.5$.

α_v	Critical values of λ_1		Ranges of λ_1 with		
	λ_{1WB}	λ^*_{1WB}	Double solutions	Single solution	No solution
–0.1	1.846944	0.000331	[1.846944, ∞) & (0,0.000331]	–	{0} & (0.000331,1.846944)
–0.2	1.589668	0.000474	[1.589668, ∞) & (0,0.000474]	–	{0} & (0.000474,1.589668)
–0.3	1.288645	0.000537	[1.288645, ∞) & (0,0.000537]	–	{0} & (0.000537,1.288645)
–0.4	0.903521	0.001097	[0.903521, ∞) & (0,0.001097]	–	{0} & (0.001097,0.903521)
–0.45	0.635811	0.001493	[0.635811, ∞) & (0,0.001493]	–	{0} & (0.001493,0.635811)
–0.49	0.283157	0.003572	[0.283157, ∞) & (0,0.003572]	–	{0} & (0.003572,0.283157)
–0.499	0.088714	0.012104	[0.088714, ∞) & (0,0.003572]	–	{0} & (0.003572,0.088714)
–0.5	–	–	–	(0, ∞)	{0}
–0.6	–	–	–	(0, ∞)	{0}
–0.9	–	–	–	(0, ∞)	{0}

σ is electrical conductivity of fluid, B denotes applied magnetic field strength, ρ is fluid density.

The boundary conditions are

$$u = -U_w(x) + L\frac{\partial u}{\partial y},\ \ v = 0 \text{ at } y = 0, \tag{2.3}$$

$$u \to 0, \frac{\partial u}{\partial y} \to 0 \text{ as } y \to \infty, \tag{2.4}$$

where $U_w(= cx)$ is contracting velocity of flat sheet (with $c > 0$ is constant) and L represents slip length. The second condition in (2.4) is augmented condition, since considered flow is on contracting flat sheet which is a boundless domain, which had been already introduced by Garg and Rajagopal.[66] A complete sketch of flow model may be found in Fig. 1.

3. Similarity treatment and solution

Now, following similarity transformations are introduced[41,67]:

$$\psi = \sqrt{\upsilon x U_w}\, f(\eta) \text{ and } \eta = y\sqrt{\frac{U_w}{x\upsilon}}, \tag{3.1}$$

where ψ is stream function with $u = \partial\psi/\partial y \,\&\, v = -\partial\psi/\partial x$ with η being similarity variable.

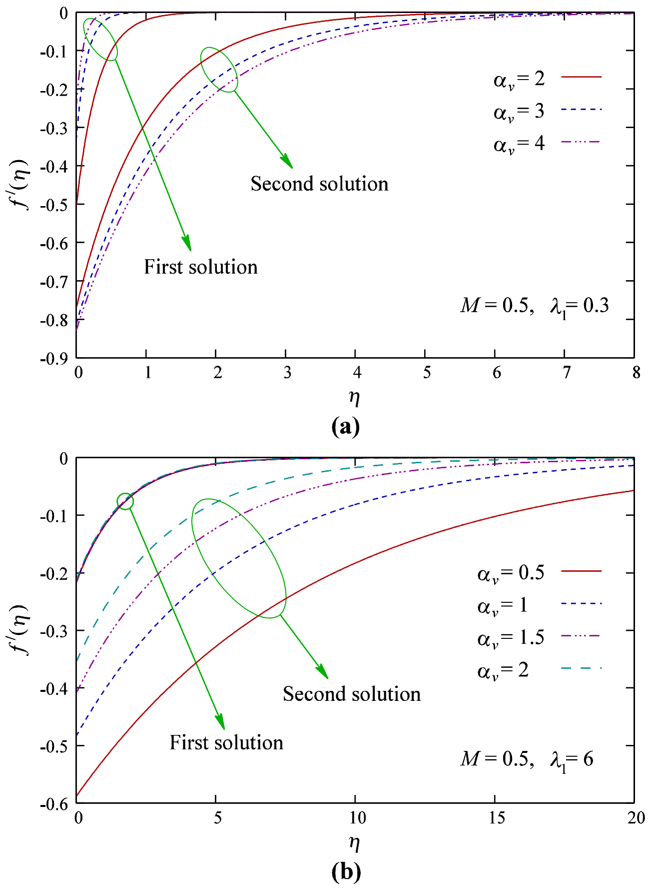


Fig. 4. Double velocity $f'(\eta)$ for various α_v with (a) $M = 0.5$ and $\lambda_1 = 0.3$, and (b) $M = 0.5$ and $\lambda_1 = 6$ (second-grade fluid, $\alpha_v > 0$).

Table 4
The critical values and ranges of λ_1 for existence of double, single and no similarity solutions for second-grade fluid with various M.

M	α_v	Critical values of λ_1		Ranges of λ_1 with		
		λ_{1WB}	λ^*_{1WB}	Double solutions	Single solution	No solution
0.3	–0.2	8.925778	0.000180	[8.925778,∞) & (0,0.000180]	–	{0} & (0.000180,8.925778)
	–0.6	8.077755	0.000278	[8.077755,∞) & (0,0.000278]	–	{0} & (0.000278,8.077755)
0.4	–0.2	4.154135	0.000258	[4.154135,∞) & (0,0.000258]	–	{0} & (0.000258,4.154135)
	–0.6	3.229882	0.000510	[3.229882,∞) & (0,0.000510]	–	{0} & (0.000510,3.229882)
0.45	–0.6	1.620880	0.000812	[1.620880,∞) & (0,0.000812]	–	{0} & (0.000812,1.620880)
0.48	–0.6	0.476263	0.002484	[0.476263,∞) & (0,0.002484]	–	{0} & (0.002484,0.476263)
0.483	–0.6	0.222681	0.005239	[0.222681,∞) & (0,0.005239]	–	{0} & (0.005239,0.222681)
0.4838	–0.6	0.060296	0.020791	[0.060296,∞) & (0,0.020791]	–	{0} & (0.020791,0.060296)
0.4839	–0.6	–	–	–	(0,∞)	{0}
0.49	–0.6	–	–	–	(0,∞)	{0}
0.5	–0.2	1.589668	0.000474	[1.589668,∞) & (0,0.000474]	–	{0} & (0.000474,1.589668)
	–0.6	–	–	–	(0,∞)	{0}
0.55	–0.2	0.404289	0.001451	[0.404289,∞) & (0,0.001451]	–	{0} & (0.001451,0.404289)
0.555	–0.2	0.124943	0.005107	[0.124943,∞) & (0,0.005107]	–	{0} & (0.005107,0.124943)
0.5555	–0.2	0.035990	0.016814	[0.035990,∞) & (0,0.016814]	–	{0} & (0.016814,0.035990)
0.5556	–0.2	–	–	–	(0,∞)	{0}
0.56	–0.2	–	–	–	(0,∞)	{0}
0.6	–0.2	–	–	–	(0,∞)	{0}
	–0.6	–	–	–	(0,∞)	{0}
0.7	–0.2	–	–	–	(0,∞)	{0}
	–0.6	–	–	–	(0,∞)	{0}
0.8	–0.2	–	–	–	(0,∞)	{0}
	–0.6	–	–	–	(0,∞)	{0}
1	–0.2	–	–	–	(0,∞)	{0}
	–0.6	–	–	–	(0,∞)	{0}
1.2	–0.2	–	–	–	[0,∞)	–
	–0.6	–	–	–	[0,∞)	–

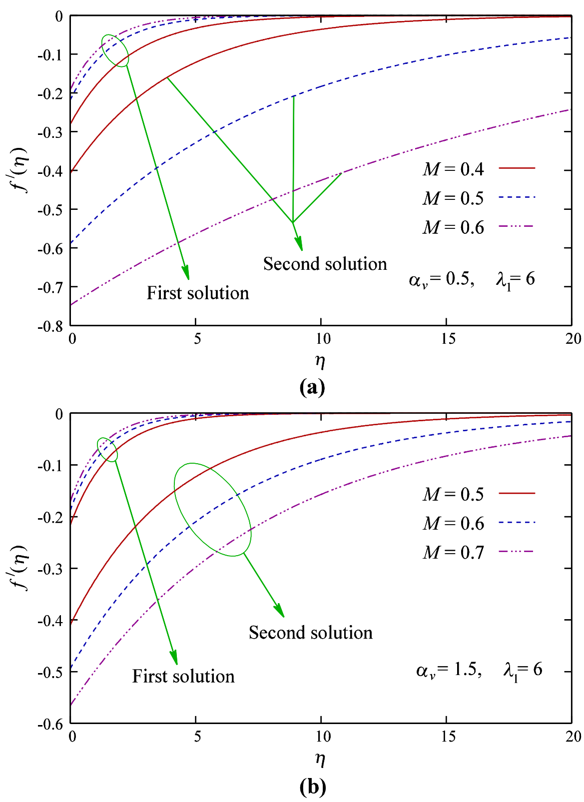


Fig. 5. Double velocity $f'(\eta)$ for various M with (a) $\alpha_v = 0.5$ and $\lambda_1 = 6$, and (b) $\alpha_v = 1.5$ and $\lambda_1 = 6$ (second-grade fluid, $\alpha_v > 0$).

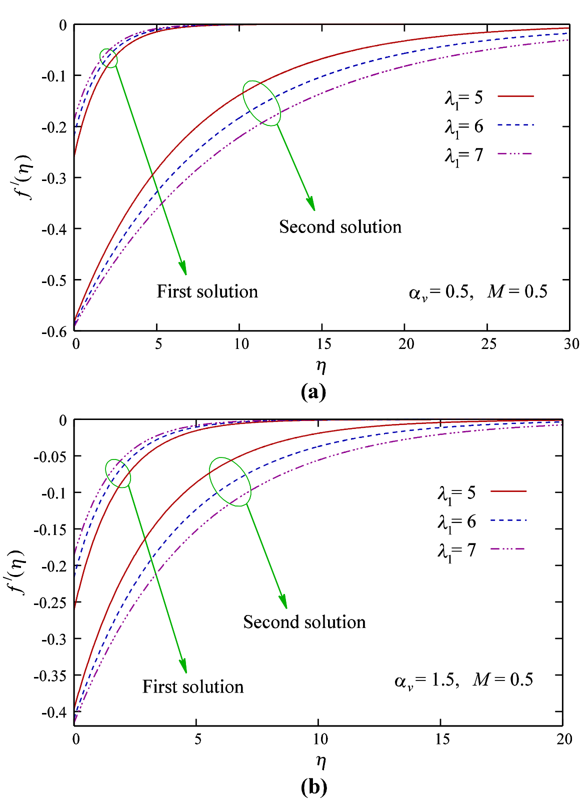


Fig. 6. Double velocity $f'(\eta)$ for various λ_1 with (a) $\alpha_v = 0.5$ and $M = 0.5$, and (b) $\alpha_v = 1.5$ and $M = 0.5$ (second-grade fluid, $\alpha_v > 0$).

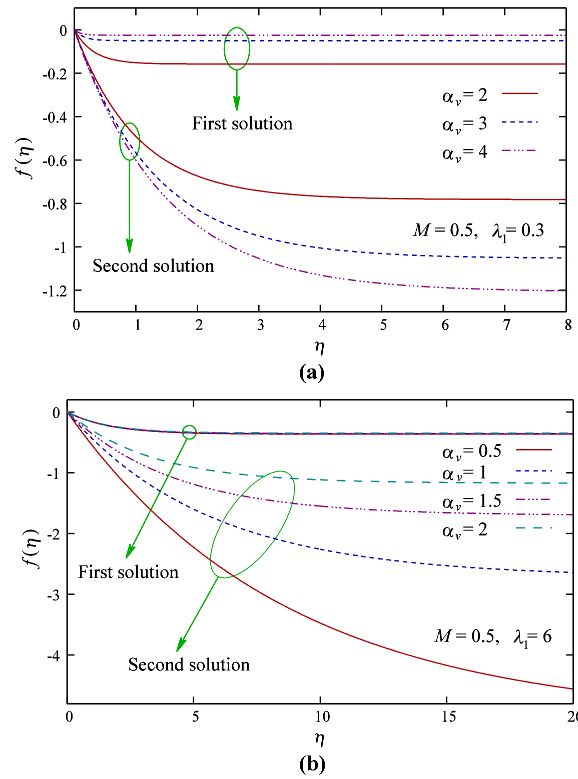


Fig. 7. Double dimensionless stream function $f(\eta)$ for various α_v with (a) $M = 0.5$ and $\lambda_1 = 0.3$, and (b) $M = 0.5$ and $\lambda_1 = 6$ (second-grade fluid, $\alpha_v > 0$).

In view of relations in (3.1) we finally obtain following self-similar equation:

$$f''' + f f'' - f'^2 - \alpha_v(f''^2 - 2f' f''' + f f'''') - M f' = 0, \tag{3.2}$$

where $\alpha_v = \pm ck^*/v$ is viscoelastic parameter with $\alpha_v > 0$ corresponding to second-grade fluid and $\alpha_v < 0$ for Walter's B fluid (with $|\alpha_v| < 1$) and $M = (\sigma B^2)/(c\rho)$ is magnetic parameter.

Boundary conditions reduce to

$$\begin{aligned} &f(0) = 0,\\ &f'(0) = -1 + \lambda_1 f''(0),\\ &f'(\infty) = 0,\\ &f''(\infty) = 0, \end{aligned} \tag{3.3}$$

where $\lambda_1 = L(c/v)^{1/2}$ is slip parameter.

The quantity related to wall drag-force, i.e., local skin-friction coefficient C_f is defined by $C_f = \frac{\tau_w}{\frac{1}{2}\rho U_w^2}$, where τ_w is wall shear stress stated as:

$$\tau_w = \rho\left[v\left(\frac{\partial u}{\partial y}\right) \pm k^*\left(2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + u\frac{\partial^2 u}{\partial x \partial y}\right)\right]_{y=0}. \tag{3.4}$$

Thus, C_f is established as: $\frac{1}{2}Re_x^{-1/2}C_f = [1 - 3\alpha_v\{1 - \lambda_1 f''(0)\}]f''(0)$, where $Re_x = U_w x/v$ denotes local Reynolds number.

A closed-form exact similarity solution of Eq. (3.2) with transformed boundary conditions (3.3) can be obtained as:

$$f(\eta) = -\frac{1}{\mu^* + \lambda_1\mu^{*2}} + \frac{1}{\mu^* + \lambda_1\mu^{*2}}e^{-\mu^*\eta}, \tag{3.5}$$

where μ^* is a positive constant which obeys the cubic equation:

$$\lambda_1\mu^{*3} + (1 - \alpha_v)\mu^{*2} - M\lambda_1\mu^* - M + 1 = 0. \tag{3.6}$$

So, there is a valid possibility of three real values of μ^* and may be for certain set of values of parameters these three values of μ^* found to be positive.

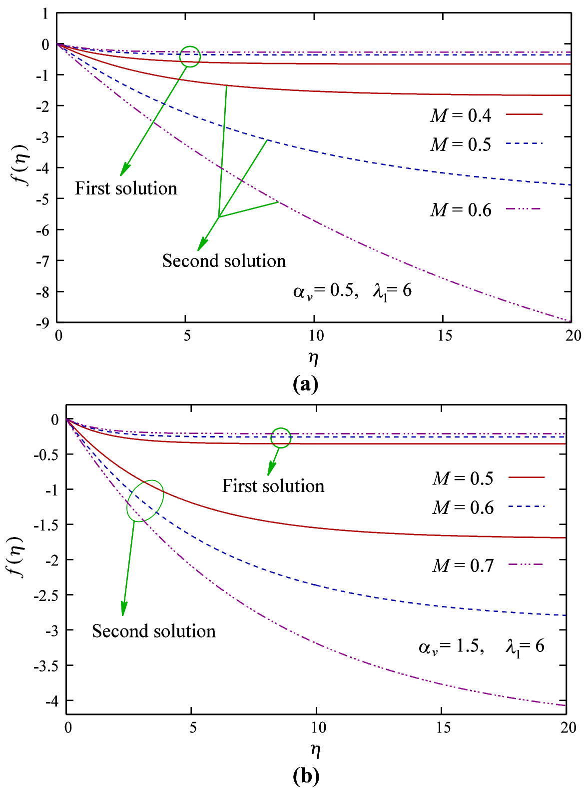


Fig. 8. Double dimensionless stream function $f(\eta)$ for various M with (a) $\alpha_v = 0.5$ and $\lambda_1 = 6$, and (b) $\alpha_v = 1.5$ and $\lambda_1 = 6$ (second-grade fluid, $\alpha_v > 0$).

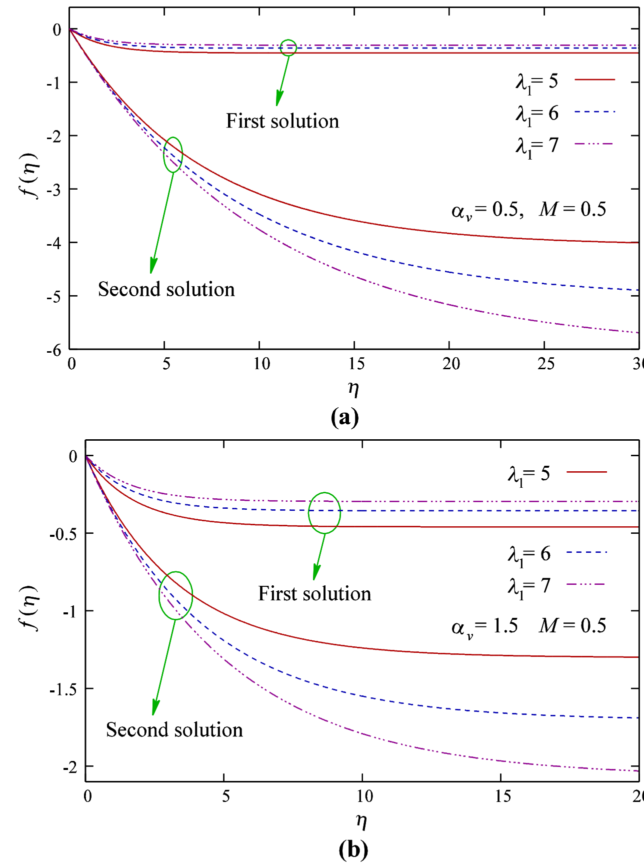


Fig. 9. Double dimensionless stream function $f(\eta)$ for various λ_1 with (a) $\alpha_v = 0.5$ and $M = 0.5$, and (b) $\alpha_v = 1.5$ and $M = 0.5$ (second-grade fluid, $\alpha_v > 0$).

Now, for $\lambda_1 = 0$, i.e., for MHD flow with usual no-slip condition at the boundary, Eq. (3.6) gives $\mu^* = \sqrt{(M-1)/(1-\alpha_v)}$ (only positive value) and the corresponding similarity solution becomes

$$f(\eta) = \frac{1}{\sqrt{(M-1)/(1-\alpha_v)}}\left(e^{-\eta\sqrt{(M-1)/(1-\alpha_v)}} - 1\right) \tag{3.7}$$

So, for no-slip boundary layer flow unique self-similar solution exists for second-grade fluid with $\alpha_v > 1$ only when $M < 1$ and with $\alpha_v < 1$ only when $M > 1$. Whereas in case of Walter's B fluid ($\alpha_v < 0$), unique self-similar solution exists only when $M > 1$.

Now, to obtain the roots of (3.6), the above cubic equation in μ^* is converted into an incomplete cubic equation as:

$$\tau^{*3} + m_1\tau^* + m_2 = 0, \tag{3.8}$$

where $\tau^* = \mu^* + \frac{1-\alpha_v}{3\lambda_1}$, $m_1 = \frac{-3\lambda_1^2 M-(1-\alpha_v)^2}{3\lambda_1^2}$ and $m_2 = \frac{2(1-\alpha_v)^3-9\lambda_1^2 M(1-\alpha_v)+27\lambda_1^2(1-M)}{27\lambda_1^3}$.

The roots of incomplete cubic Eq. (3.8) are

$\tau_1^* = c_1 + c_2$, $\tau_2^*, \tau_3^* = -\frac{1}{2}(c_1+c_2) \pm i\frac{\sqrt{3}}{2}(c_1-c_2)$,

where $c_1 = \sqrt[3]{-\frac{m_2}{2}+\sqrt{c_3}}$, $c_2 = \sqrt[3]{-\frac{m_2}{2}-\sqrt{c_3}}$, $i^2 = -1$, $c_3 = \left(\frac{m_1}{3}\right)^3 + \left(\frac{m_2}{2}\right)^2$.

So, the expression of dimensionless velocity is $f'(\eta) = -\frac{1}{1+\lambda_1\mu^*}e^{-\mu^*\eta}$ and the local skin-friction coefficient is $\frac{1}{2}Re_x^{-1/2}C_f = \left[1 - 3\alpha_v\left\{1-\lambda_1\frac{\mu^*}{1+\lambda_1\mu^*}\right\}\right]\frac{\mu^*}{1+\lambda_1\mu^*}$, where $\mu^*(>0)$ follows Eq. (3.6).

The solution (3.5) of Eq. (3.2) with (3.3) is obtained solving Eq. (3.6) in μ^* and Eq. (3.8) in τ^* for different values of α_v, λ_1 and M using a computer programme in FORTRAN.

4. Results and discussion

After getting positive values of μ^* for several values of α_v, λ_1 and M using the aforesaid programme, it reveals that in cases of second grade fluids with α_v being positive, double similarity solutions exist if slip parameter λ_1 equals to or exceeds a certain critical value, say, λ_{1SG} and if λ_1 is below λ_{1SG} then no similarity solution occurs and it is a similar situation like Newtonian case($\alpha_v = 0$). The values of λ_{1SG} are 2.477049, 2.985366 and 3.674236 for $\alpha_v = 0.2$, 0.5 and 1, respectively with $M = 0.5$. For $\alpha_v > 1$, two ranges of λ_1 where double solutions exist and it means that these exist two critical values of λ_1, say, λ_{1SG} and λ_{1SG}^*. So, double similarity solutions occur if λ_1 equals to or exceeds λ_{1SG} and λ_1 less or equals to λ_{1SG}^*, whereas no solution exists for λ_1 being between the two values. The values of λ_{1SG} and λ_{1SG}^* for $\alpha_v = 1.1$, 1.5, 2, 3 are respectively $\lambda_{1SG} = 3.794036$, 4.223462, 4.657540, 5.167583 and $\lambda_{1SG}^* = 0.017216$, 0.193323, 0.554367, 1.672153. One can easily varies that the gap narrows down with rising α_v. So, superiority of elastic property over viscous property of fluid results extension of solution existence range of λ_1. Furthermore, if α_v grows up to and above 4, then in whole region double solutions exist, i.e., for each non-negative value of λ_1 double solutions appears in existence and this innovative fact happens for $M = 0.5$, which are described through Table 1 in details. Now, obviously next attention will be focused on the impact of magnetic parameter M on the solution existence ranges of λ_1 for second-grade fluids. For the increment of M, double solutions range of λ_1 elongates; it means that the values of λ_{1SG} falls down(when $\alpha_v = 0.5 < 1$) and suddenly, when $M > 0.75$ solution find out to be unique for each positive λ_1. Also, for $\alpha_v = 1.5 > 1$, when $M \geq 0.8572$, double solutions are achieved for any positive λ_1(with unique solution for $\lambda_1 = 0$) and finally when $M \geq 1.5$, then solution confirms

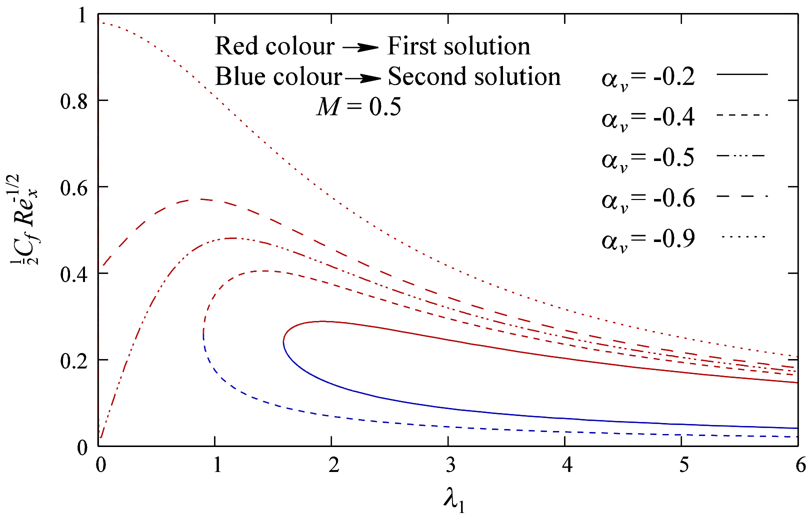
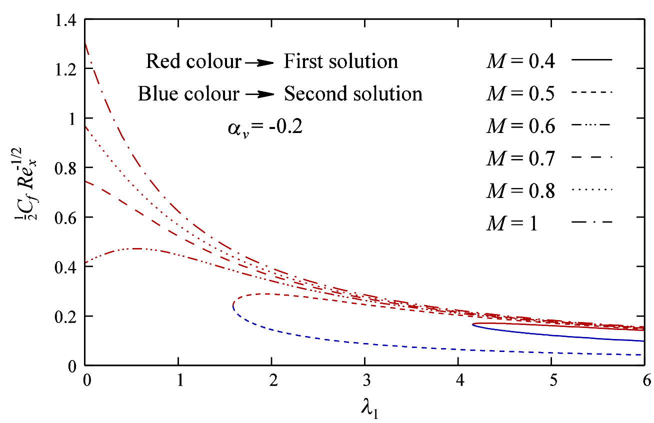


Fig. 10. $\frac{1}{2}Re_x^{-1/2}C_f$ vs. λ_1 for various α_v (Walter's B fluid, $\alpha_v < 0$) with $M = 0.5$.

its uniqueness for slip flow. Details of values of λ_{1SG} and λ^*_{1SG} and different solution ranges for several M are described in Table 2. So, it is worth noting to state that introduction of magnetic field of sufficiently strong strength able to stabilize the uncertainty situation of solution existence and is resulting unique similarity solution for slip flow over contracting sheet. Hence, the magnetic field is able to control the vorticity formed due to contraction of the sheet. Also, the skin-friction sketches confirming double, single and no solutions are demonstrated in Figs. 2 and 3 and double velocity and dimensionless stream function are depicted in Figs. 4–9 for various α_v, λ_1 and M. For growth of α_v surface drag reduces, whereas with M the drag enhances for higher slip value (λ_1) for first branch for second-grade fluid. The dominancy of elastic behaviour on the viscosity, boundary layer thickness (BLT) becomes thinner (Fig. 4) in first solution. Also, magnetic field works against the transport phenomenon and expected outcome is witnessed in Fig. 5, i.e., the BLT reduces for first solution and similar impact for boundary slip is observed (Fig. 6).

Next, for Walter's B fluid ($\alpha_v < 0$ with $|\alpha_v| < 1$) double similarity solutions exist in two ranges of λ_1, i.e., there exists two critical values, say, λ_{1WB} and λ^*_{1WB} and if λ_1 goes above and below (with equality) to those critical values, then aforesaid double solutions occurs. Also, there is no solution when λ_1 have values in between those critical values along with no-slip case ($\lambda_1 = 0$). The values of λ_{1WB} and λ^*_{1WB} for $\alpha_v = -0.2, -0.4, -0.499$ are respectively $\lambda_{1WB} = 1.589668, 0.903521, 0.088714$ and $\lambda^*_{1WB} = 0.000474, 0.001097, 0.012104$. This character abolishes when $\alpha_v \le -0.5$ and double solutions reduce to single one having existence range as all positive values of λ_1, i.e., for any type of slip flow. While discussing aforesaid characters of Walter's B fluid magnetic parameter is supposed as 0.5 and these all are recorded in Table 3. Impacts of magnetic field on solution existence range and uniqueness of solution are prescribed through Table 4 for two values of α_v(-0.2 and -0.6). It reveals that for any value of α_v unique slip flow similarity solution exists after rise of M up to certain level and even for no-slip flow for $M > 1$. Similar to second-grade case, for Walter's B case magnetic field shows identical impact, i.e., strong MHD effect destroys the uncertainty of existence of similarity solution and makes it unique for slip flows as well as no-slip flow. Complete structures of skin-friction for different involved parametric values are established through Figs. 10 and 11. Whereas in Figs. 12–13, velocity and dimensionless stream function are portrayed for several α_v, λ_1 and M. The surface drag rises with magnitude of α_v and M first branch for Walter's B case. Identical influences to that of the second grade for viscoelastic, magnetic and slip parameters is detected.



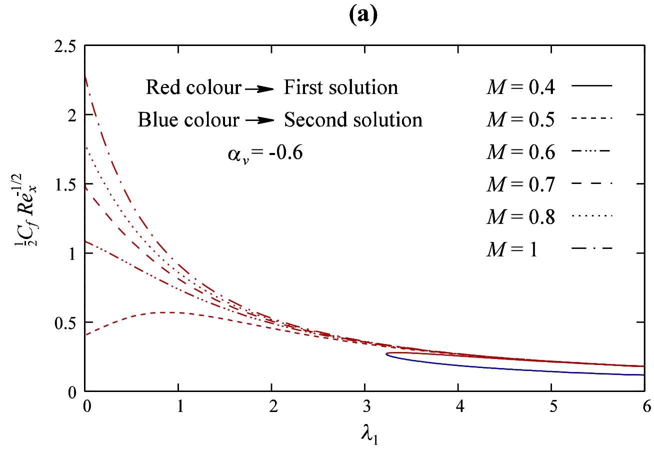


Fig. 11. $\frac{1}{2}Re_x^{-1/2}C_f$ vs. λ_1 for various M with (a) $\alpha_v = -0.2$ and (b) $\alpha_v = -0.6$ (Walter's B fluid, $\alpha_v < 0$).

5. Final conclusions

The study of existence of multiple exact similarity solutions for MHD flows of second-grade and Walter's B fluids due continuously contracting flat sheet with non-convectional slip condition is performed. After

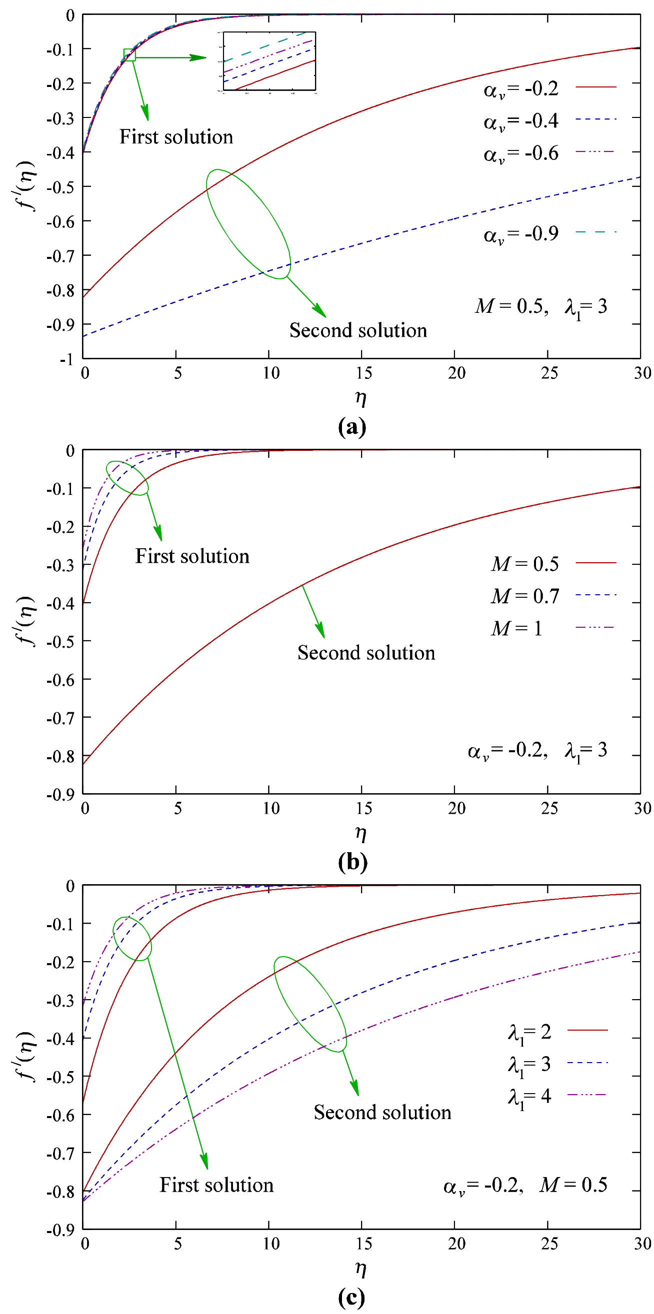


Fig. 12. Double and unique velocities $f'(\eta)$ for various (a) $\alpha_v(M = 0.5$ and $\lambda_1 = 3)$, (b) M ($\alpha_v = -0.2$ and $\lambda_1 = 3$) and (c) λ_1 ($\alpha_v = -0.2$ and $M = 0.5$) (Walter's B fluid, $\alpha_v < 0$).

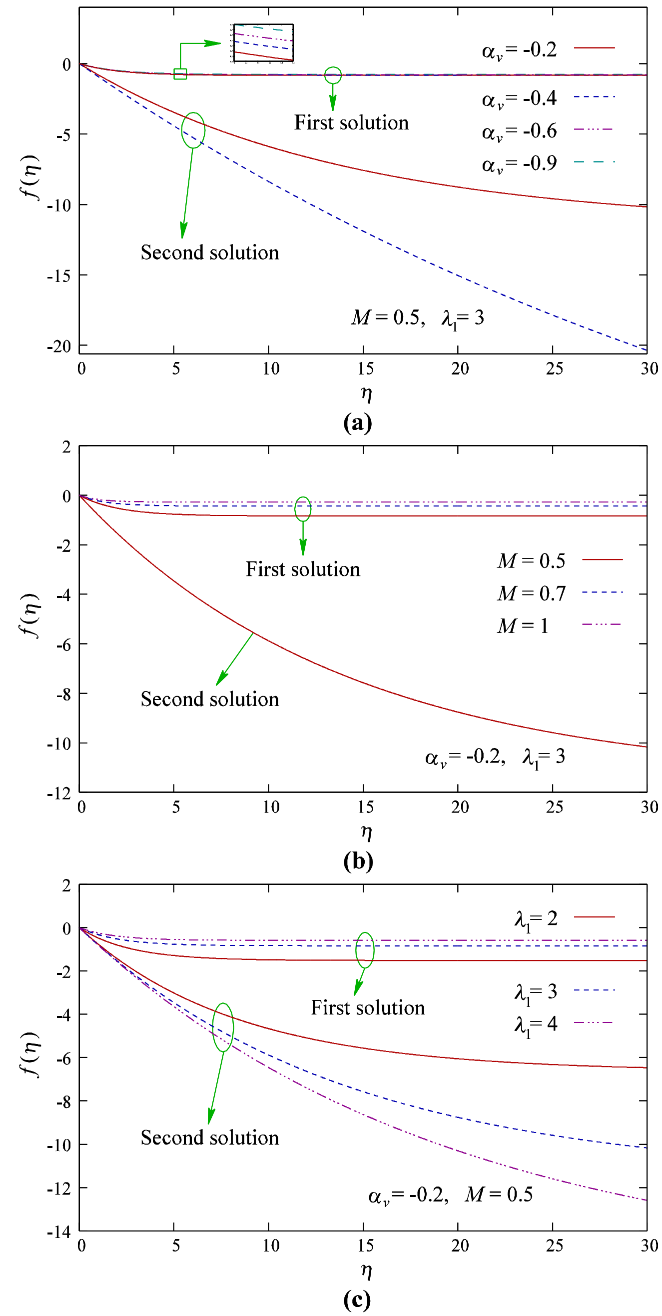


Fig. 13. Double and unique dimensionless stream functions $f(\eta)$ for various (a) α_v ($M = 0.5$ and $\lambda_1 = 3$), (b) M ($\alpha_v = -0.2$ and $\lambda_1 = 3$) and (c) λ_1 ($\alpha_v = -0.2$ and $M = 0.5$) (Walter's B fluid, $\alpha_v < 0$).

getting similarity solutions of converted flow governing equation and analysing those, the following important remarks can be made:

(a) For the flow of both viscoelastic type fluids, double solutions exist for certain set of values of involved parameters.
(b) In cases of second-grade fluid, for $\alpha_v < 1$ double solutions exist when slip parameter exceeds certain critical value. But for $\alpha_v > 1$, there are two ranges where dual solutions exist and as $\alpha_v(\geq 4)$ increases double solutions exist for all non-negative values of slip parameter (with $M = 0.5$).
(c) If elastic property is dominant over viscous property of second-grade fluid, then solution existence range of λ_1 is extended.
(d) Whereas for Walter's B fluid, with smaller magnitude of $\alpha_v(< 0.5)$ there are two ranges of double solutions and when $|\alpha_v| \geq 0.5$ solution becomes unique (with $M = 0.5$).
(e) Also, introduction of magnetic field ends the uncertainty of solution existence for various λ_1 for both fluids and for sufficiently strong magnetic field, there exists only one solution in all cases.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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