JCIS Open 8 (2022) 100064

Contents lists available at ScienceDirect

JCIS Open

journal homepage: www.journals.elsevier.com/jcis-open

Unsteady axisymmetric flow of nanofluid on nonlinearly expanding surface with variable fluid properties

Sohita Rajput [a], Krishnendu Bhattacharyya [a,*], Amit Kumar Pandey [a], Ali J. Chamkha [b]

[a] Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, 221005, Uttar Pradesh, India
[b] Faculty of Engineering, Kuwait College of Science and Technology, Doha District, Kuwait

ARTICLE INFO

Keywords:
Axisymmetric flow
Unsteady nanofluid
Nonlinear expanding sheet
Variable viscosity
Variable thermal conductivity

ABSTRACT

The physical phenomena of nanofluid at high temperature motivate us to analyze problems with temperature-reliant fluid properties, like viscosity and thermal conductivity. Since in glass blowing, viscosity and thermal conductivity of the fluid may gets affected in such high temperature. This communication deals with the unsteady flow of nanofluid generated by nonlinear expansion of the surface. Temperature-dependent fluid viscosity and thermal conductivity are considered in the investigation of the problem. The flow of nanofluid is modeled using famous the *Buongiorno's two-phase* model, which includes the simultaneous effect of Brownian motion and thermophoresis diffusion. Appropriate transformations are adopted to obtain the ODEs from governing PDEs. Then MATLAB 'bvp4c' computation is used to solve the problem and to get a clear insight of the influences of various parameters. Graphical comparisons are made to check the accuracy of used numerical method. The study explores that heat transfer rate significantly enhances by the index of nonlinearity, variable viscosity and thermal conductivity parameters. Unsteadiness of the flow can be used as a controlling parameter to reduce the surface drag, heat and nano-mass transfer rate. Variable viscosity parameter leads to enhance the velocity near the surface and reducing the concentration of the nanoparticles. The thermal and concentration boundary layer thickens with thermal conductivity parameters. Nanofluid temperature and concentration of nanoparticles decay with nonlinear expanding index.

1. Introduction

The flow of conventionally working fluid (water, oil, air, biofluids, lubricants and polymer solutions, etc.) is widely used in our daily life applications, such as machinery manufacturing, air conditioning, chemical production, automobiles and so on. Due to advancements in technology these traditional fluids are not adequate to work as heat transfer media to fulfill the demanding requirements of a high heat transfer rate. Nanotechnology is an innovative idea in this direction. Nanotechnology is the study of material at nano-scale along with its production and applications. The aim of research in nanotechnology mainly focuses to improve the quality of products, saving energy, to make products economically convenient, enhance the equipment's life and to reduce the time of processing. An alternate way to overcome the limited use of lower thermally conductive working fluids is by dispersing nanometer-sized metallic/non-metallic solid particles in it. These particles are known as nanoparticles and a mixture of base fluid with suitable nanoparticles is termed as nanofluid. This advanced concept is given by Choi [1]. Nanofluids have applications in a broad domain depending on several parameters, such as shape, size, type and concentration of nanoparticles in many industries [2]. Analysis of transfer characteristics of nanofluid can be figured out by two proposed models. One of them is Tiwari and Das [3] model, based on homogeneous principle and the other is the *Buongiorno* [4] model based on slip mechanism between nanoparticles and conducting fluid. *Buongiorno* [4] proposed a two-component and four-equations model which considers the significance of the Brownian and Thermophoresis mechanism over other slip mechanisms. Numerous studies have been performed based on above two models [3,4] with various aspects to examine the different characteristics of nanofluid flows [5-10]. Recently, non-similar solution of EMHD flow of Casson nanofluid with various shapes of nanoparticles is analyzed by Hussain et al. [11], and Ajeet at al [12]. discussed the significant role of radius of copper nanoparticles immersed in water for time-dependent flow in presence of porous medium.

Analysis of axisymmetric flow caused by radially moving surface has motivated many researchers to focus on such flows due to its wide range of uses in industrial areas, such as rapid spray cooling, polymer

* Corresponding author.
E-mail addresses: sohita.math@gmail.com (S. Rajput), krishmath@bhu.ac.in, krish.math@yahoo.com (K. Bhattacharyya).

https://doi.org/10.1016/j.jciso.2022.100064
Received 28 June 2022; Received in revised form 23 August 2022; Accepted 29 September 2022

Nomenclature

a	a constant
A	unsteady parameter
A_1	a constant
C	concentration of nanoparticles
C_{fr}	local skin-friction coefficient
C_w	concentration at the wall
C_∞	concentration in free-stream
D_B	Brownian diffusion coefficients
D_T	thermophoretic diffusion coefficients
f	dimensionless stream function
m	nonlinear expanding parameter
Nb	Brownian motion parameter
Nt	thermophoresis parameter
Nu_r	local Nusselt number
Pr	Prandtl number
q_m	wall mass flux
q_w	wall heat flux
r	distance from the origin
(r, Θ, z)	cylindrical coordinate system
Re_r	local Reynolds number
Sc	Schmidt number
Sh_r	local Sherwood number
t	time
T	temperature of nanofluid
T_w	temperature at the wall
T_∞	temperature in free-stream
T_r	a constant
u, w	velocity components in the r- and z-directions, respectively
u_w	velocity at the wall
V	velocity field

Greek symbols

ε	thermal conductivity parameter
η	a variable
γ	thermal property of the nanofluid
θ	dimensionless temperature
θ_r	variable viscosity parameter
κ	thermal conductivity
κ_∞	thermal conductivity at ambient temperature
λ	a constant
μ	viscosity
μ_∞	dynamic viscosity at ambient temperature
ρ	density
(ρC_p)	heat capacity
υ	kinematic viscosity
φ	dimensionless concentration
τ	ratio of heat capacities
τ_w	wall shear stress

extraction, glass blowing and metal foundries. In 2D steady case, axisymmetric boundary layer equations over a continuous cylindrical solid surface initiated by Sakiadis [13] in 1961. A problem of axisymmetric flow for second-grade fluid due to the radially expanding surface was reported by Ariel [14]. Sajid et al. [15] conducted the work for axisymmetric flow for third-grade fluid and solved the problem via HAM. Later, Shahzad et al. [16] performed an analytical calculation on axisymmetric MHD flow past a nonlinear stretching surface for the index of nonlinearity taken as 3 and presented correlations for velocity and temperature profiles as functions of the incomplete gamma function. In literature, there are several studies regarding nonlinear expanding sheets, but very few have pointed out the radially nonlinear expanding sheet. Closed-form solution of the MHD axisymmetric flow problem of hybrid nanofluid with permeable stretching/shrinking sheet with assuming thermal radiation effect was provided by Khan et al. [17]. Recently, many researchers inspected the axisymmetric boundary layer problems either analytically or numerically [18-23].

In served research works on the flow and heat transfer problems ignore the variation of physical properties with temperature, but to predict the rate of heat transfer more precisely it is important to consider the change of viscosity and thermal conductivity with temperature. The problems of this kind have relevant applications in glass fiber production, paper production, geothermal energy extraction, underground storage system and nuclear reactor cooling. Many renowned researchers have introduced several temperature-based viscosity models. Out of which some important viscosity-temperature equations are proposed by Reynold, Vogel (discussed by Ellahi et al. [24]) and Ling and Dybbs [25]. Here, Ling and Dybbs [25] viscosity-temperature equation is more appropriate for a wide range of temperature. While in Reynold's viscosity model, there is a restriction that it is accurate only for a confined range of temperature, and on the other hand, Vogel's viscosity model is more useful to model the third-grade fluid as discussed by Knežević et al. [26] and Das et al. [27]. For the thermal conductivity-temperature equation, variation of thermal conductivity with temperature is usually assumed in a linear manner (see Kay [28]). Further, the above mentioned relations for variable viscosity and thermal conductivity have been considered by several scientists [29-35] for modeling the thermophysical properties of the flows. Furthermore, studies of nanofluid with temperature varying thermal properties are performed by Khan et al. [36] and Lawal et al. [37] for Walter's B and Eyring–Powell nanofluids, respectively. Akbar et al. [38] formulated the unsteady nanofluid flow problem with variable fluid properties and consequently found the exact solution of this flow problem.

In several practical problems in industries, a steady-state radially expanding surface with linear velocity is not enough for the accurate prediction of physical phenomena of the flow and heat transfer when there are nanoparticles in the flow field and fluid properties are considered temperature-dependent. The practical relevance of unsteadiness in flow due to radial expansion of flat sheet with nanoparticles and its nonlinear expansion velocity acts to create the motivation towards the explanation of an unsteady problem involving radial expansion with nonlinear velocity, variable fluid properties and nanoparticles simultaneously. So, in the present study the boundary layer flow of nanofluid over the unsteady radially nonlinear expanding surface along with temperature-dependent viscosity and thermal conductivity is examined. Four equations two-phase *Buongiorno* model is adopted to formulate the mathematical description of the boundary layer flow problem. The novel feature regarding this study is to determine the significance of implementing variable viscosity model(Ling and Dybbs [25]) and variable thermal conductivity linear temperature-conductivity model [29] to know the heat and nano-mass transfer characteristics of nanofluid fluid flow due to nonlinear radially expanding surface with Brownian and thermophoresis slip mechanisms. Then governing partial differential equations(PDEs) are converted into ordinary differential equations(ODEs) by using some suitable transformations and then those are converted in the system of first-order ODEs. Later, the system is numerically solved by the 'bvp4c' method (a MATLAB programme). For validation of the obtained solution, we compare our findings with the exact solutions graphically. In this investigation, the following research queries have been tried to address:

- What are the simultaneous effects of unsteadiness and nonlinear expanding of sheet in presence of nanoparticles ?
- How is the aforesaid flow field of nanofluid influenced by using temperature-reliant viscosity and thermal conductivity ?
- What are the consequences of Brownian and thermophoresis effects in unsteady nanofluid flow on radially expanding surface ?

This type problem has relevant applications in high heat transfer processes in industries and engineering, like cooling of nuclear reactor, paper manufacturing, polymer extraction, etc.

2. Flow analysis

2.1. Model development

The axisymmetric unsteady radial flow of nanofluid with temperature-dependent fluid properties over a nonlinear radially expanding sheet is considered. The flow is generated due to the nonlinear radial velocity of the sheet i.e. $u_w = ar^m(1-\lambda t)^{-1}$ where r is the distance from the origin, m is a nonlinear expanding parameter, a and λ are constants (>0) having dimension $time^{-1}$. A cylindrical polar coordinate system (r, Θ, z) is adopted for the mathematical description of the problem. The temperature and concentration are assumed constant at the surface and in the free stream, which are T_w, C_w, T_∞ and C_∞, respectively with $T_w > T_\infty$ and $C_w > C_\infty$. For time-dependent axisymmetric two-dimensional flow, velocity, temperature and nanoparticle concentration fields are taken in the following manner:

$$V = [u(r,z,t), 0, w(r,z,t)], T = T(r,z,t), C = C(r,z,t) \tag{1}$$

Under assumptions of the boundary layer structure, the governing flow, energy and nanoparticles concentration equations can be expressed as [18,31]:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu(T)\frac{\partial u}{\partial z}\right) \tag{3}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \frac{1}{\rho C_p}\frac{\partial}{\partial z}\left(\kappa(T)\frac{\partial T}{\partial z}\right) + \tau\left[D_B\frac{\partial C}{\partial z}\frac{\partial T}{\partial z} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right] \tag{4}$$

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial r} + w\frac{\partial C}{\partial z} = D_B\frac{\partial^2 C}{\partial z^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2} \tag{5}$$

where u and w are velocity components along r and z directions, respectively, T and C are nanofluid temperature and concentration of nanoparticles, ρ is density of nanofluid, μ is viscosity of nanofluid, ρC_P is heat capacity of nanofluid, κ is thermal conductivity of nanofluid, τ is ratio of heat capacity of nanoparticles to heat capacity of fluid, D_B and D_T are Brownian and thermophoretic diffusion coefficients.

The appropriate boundary conditions of the problem are given by

$$\left.\begin{aligned} &u = u_w = \frac{ar^m}{1-\lambda t}, w = 0, T = T_w, C = C_w \quad \text{at } z = 0, \\ &u \to 0, T \to T_\infty, C \to C_\infty \quad \text{as } z \to \infty. \end{aligned}\right\} \tag{6}$$

2.2. Introduction of variable fluid properties

To estimate the flow and heat transfer more precisely, variation of viscosity and thermal conductivity with temperature becomes more important in several engineering processes, such as in machinery lubrication, where viscosity and thermal conductivity of lubricants used in bearings gets affected by frictional temperature. The model used by Ling and Dybbs [25] is taken for viscosity variation estimation of the fluid, which is a good approximation for liquids, such as water and crude oil and is given by

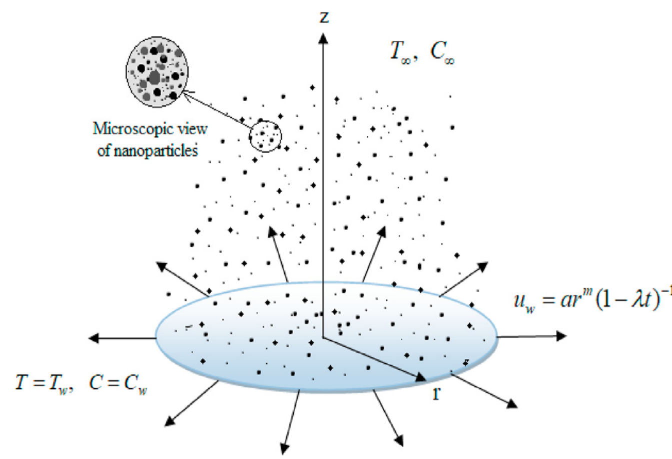


Fig. 1. Physical sketch of the problem.

$$\frac{1}{\mu} = \frac{1}{\mu_\infty}\{1 + \gamma(T_w - T_\infty)\} \tag{7}$$

where γ is the thermal property of the nanofluid (a constant with dimension $temp^{-1}$), μ_∞ is the dynamic viscosity at ambient temperature. Eq. (7) can be rearranged as

$$\frac{1}{\mu} = A_1(T - T_r) \tag{8}$$

where $A_1 = \frac{\gamma}{\mu_\infty}$ and $T_r = T_\infty - \frac{1}{\gamma}$ are constants. For the model of variable thermal conductivity, it follows (Chaim [29]):

$$\kappa = \kappa_\infty\left(1 + \varepsilon\frac{T - T_\infty}{T_w - T_\infty}\right) \tag{9}$$

where κ_∞ is the thermal conductivity away from the surface, ε is the thermal conductivity parameter. A diagram of the physical problem is portrayed in Fig. 1.

2.3. Introduction of suitable transformations

The following transformations (Mustafa et al. [18]) introduce for conversion of PDEs into ODEs:

$$\left.\begin{aligned} &u = u_w f'(\eta), w = -\sqrt{\frac{av}{(1-\lambda t)}}\left[\left(\frac{m+3}{2}\right)f + \left(\frac{m-1}{2}\right)\eta f'\right]r^{(m-1)/2}, \\ &\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \varphi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \quad \text{and} \quad \eta = r^{(m-1)/2} z\sqrt{\frac{a}{v(1-\lambda t)}}, \end{aligned}\right\} \tag{10}$$

where η is similarity variable, v is kinematic viscosity. The continuity equation (Eq. (2)) is identically satisfied by u and w (given in Eq. (10)). The dimensionless temperature can be written as

$$\theta = \frac{T - T_r}{T_w - T_\infty} + \theta_r \tag{11}$$

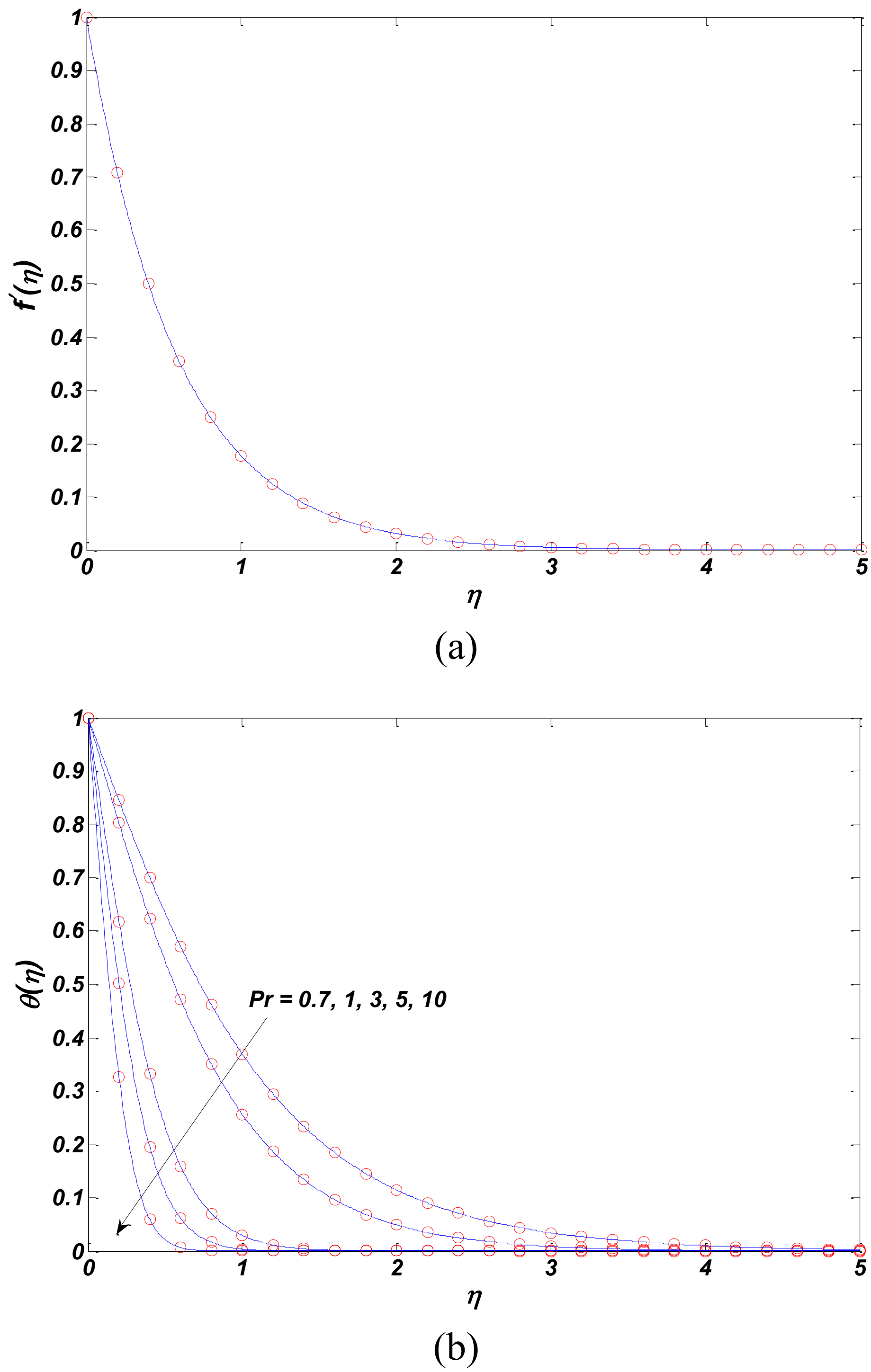


Fig. 2. Comparison of (a) velocity profiles: Numerical solution (Solid line) and exact solution (Circles) given by Eq. (19) and (b) temperature profiles for different values of Pr: Numerical solution (Solid line) and exact solution (Circles) given by Eq. (20).

where $\theta_r = \frac{T_r - T_\infty}{T_w - T_\infty} = -\frac{1}{\gamma(T_w - T_\infty)}$ is the variable viscosity parameter. Using Eq. (11), one may obtain from Eq. (8) $\mu = \mu_\infty\left(\frac{\theta_r}{\theta_r - \theta}\right)$ and from Eq. (9) $\kappa = \kappa_\infty(1+\varepsilon\theta)$.

In view of Eqs. (10) and (11), the following ODEs are obtained:

$$\left(\frac{\theta_r}{\theta_r-\theta}\right)f''' + \left(\frac{m+3}{2}\right)ff'' - mf'^2 + \left\{\frac{\theta_r}{(\theta_r-\theta)^2}\right\}f''\theta' - A\left(f' + \frac{\eta}{2}f''\right) = 0 \quad (12)$$

$$(1+\varepsilon\theta)\theta'' + \varepsilon\theta'^2 + \mathrm{Pr}\left(1-\frac{\theta}{\theta_r}\right)(1+\varepsilon\theta)\left\{\left(\frac{m+3}{2}\right)f\theta' - \frac{\eta}{2}A\theta' + Nb\theta'\varphi' + Nt\theta'^2\right\} = 0 \quad (13)$$

$$\varphi'' - Sc\left\{\frac{\eta}{2}A\varphi' - \left(\frac{m+3}{2}\right)f\varphi'\right\} + \frac{Nt}{Nb}\theta'' = 0 \quad (14)$$

where A is local unsteady parameter, Pr is Prandtl number, Nb is Brow-

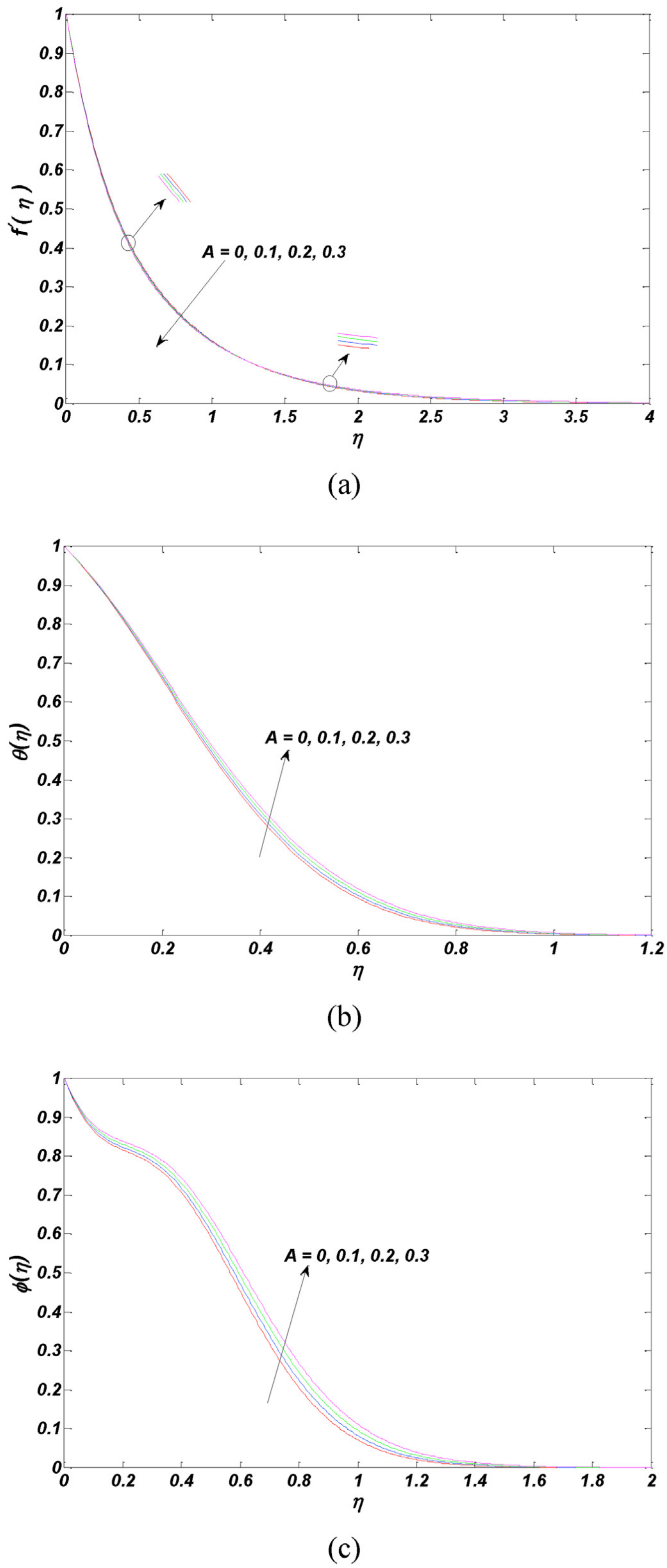


Fig. 3. Effects of A on (a) $f'(\eta)$, (b) $\theta(\eta)$ and (c) $\varphi(\eta)$.

nian motion parameter, Nt is thermophoresis parameter, Sc is Schmidt number, which are defined as:

$$A=\frac{\lambda}{ar^{m-1}},\quad \Pr=\frac{\mu C_p}{\kappa},\quad Nb=\frac{\tau D_B(C_w-C_\infty)}{\upsilon},\quad Nt=\frac{\tau D_T(T_w-T_\infty)}{T_\infty\upsilon},\quad Sc=\frac{\upsilon}{D_B}$$

The converted non-dimensional forms of the boundary conditions are

$$\left.\begin{array}{l} f(0)=0, f'(0)=1, \theta(0)=1, \varphi(0)=1, \\ f'(\infty)\to 0, \theta(\infty)\to 0, \varphi(\infty)\to 0. \end{array}\right\} \tag{15}$$

2.4. Quantities of engineering interest

The important mechanisms of the flow are local skin-friction coefficient, local Nusselt number and local Sherwood number, which are written as:

$$C_{fr}=\frac{\tau_w}{\rho u_w^2},\quad Nu_r=\frac{rq_w}{\kappa_\infty(T_w-T_\infty)} \text{ and } Sh_r=\frac{rq_m}{D_B(C_w-C_\infty)} \tag{16}$$

where τ_w, q_w and q_m are the wall shear stress, wall heat flux and wall mass flux respectively, having the following expressions:

$$\tau_w=\mu\left(\frac{\partial u}{\partial z}\right)_{z=0},\quad q_w=-\kappa\left(\frac{\partial T}{\partial z}\right)_{z=0} \text{ and } q_m=-D_B\left(\frac{\partial C}{\partial z}\right)_{z=0.} \tag{17}$$

Using Eqs. (10), (11) and (18) in Eq. (16), the surface drag force, heat and nano-mass transfer rates reduce in the following forms:

$$\left.\begin{array}{l} \mathrm{Re}_r^{1/2}C_{fr}=\left[\dfrac{\theta_r}{\theta_r-\theta(0)}\right]\left(\dfrac{d^2f}{d\eta^2}\right)_{\eta=0}, \\ \mathrm{Re}_r^{-1/2}Nu_r=-\{1+\varepsilon\theta(0)\}\left(\dfrac{d\theta}{d\eta}\right)_{\eta=0}, \\ \mathrm{Re}_r^{-1/2}Sh_r=-\left(\dfrac{d\varphi}{d\eta}\right)_{\eta=0}, \end{array}\right\} \tag{18}$$

where $\mathrm{Re}_r=\frac{ru_w}{\upsilon}$ is the local Reynolds number.

2.5. Exact solution for m = 3

In the case of fluid flow without nanoparticles, closed-form of the analytic solution of the Eqs. (12) and (13) along with boundary conditions Eq. (15) can be obtained for $m=3$ with constant properties (details given by Shahzad et al. [16]):

$$f(\eta)=\frac{1}{\sqrt{3}}\left(1-e^{-\sqrt{3}\eta}\right), \tag{19}$$

$$\theta(\eta)=\frac{\Gamma(\Pr,0)-\Gamma\left(\Pr,e^{-\sqrt{3}\eta}\Pr\right)}{\Gamma(\Pr,0)-\Gamma(\Pr,\Pr)}, \tag{20}$$

where $\Gamma(a,x)$ is the incomplete Gamma function.

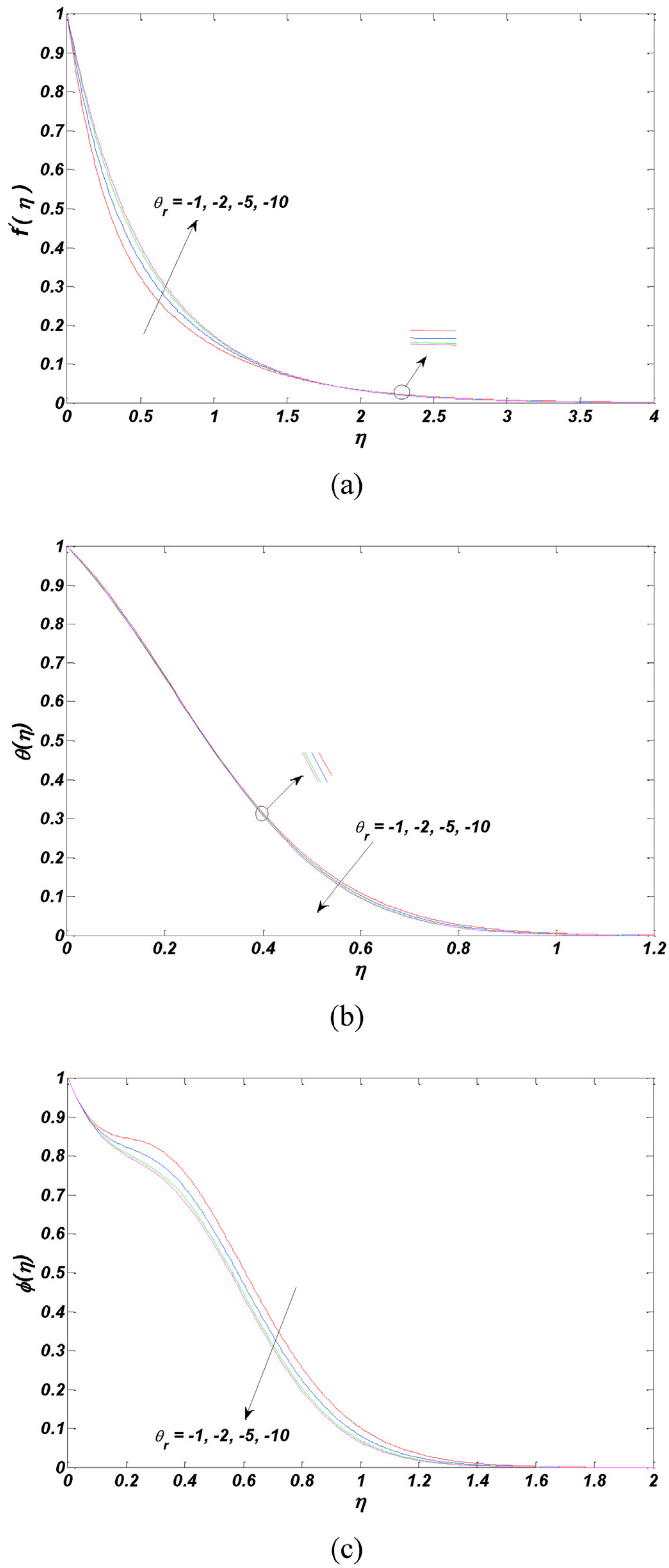


Fig. 4. Effects of θ_r on (a) $f'(\eta)$, (b) $\theta(\eta)$ and (c) $\varphi(\eta)$.

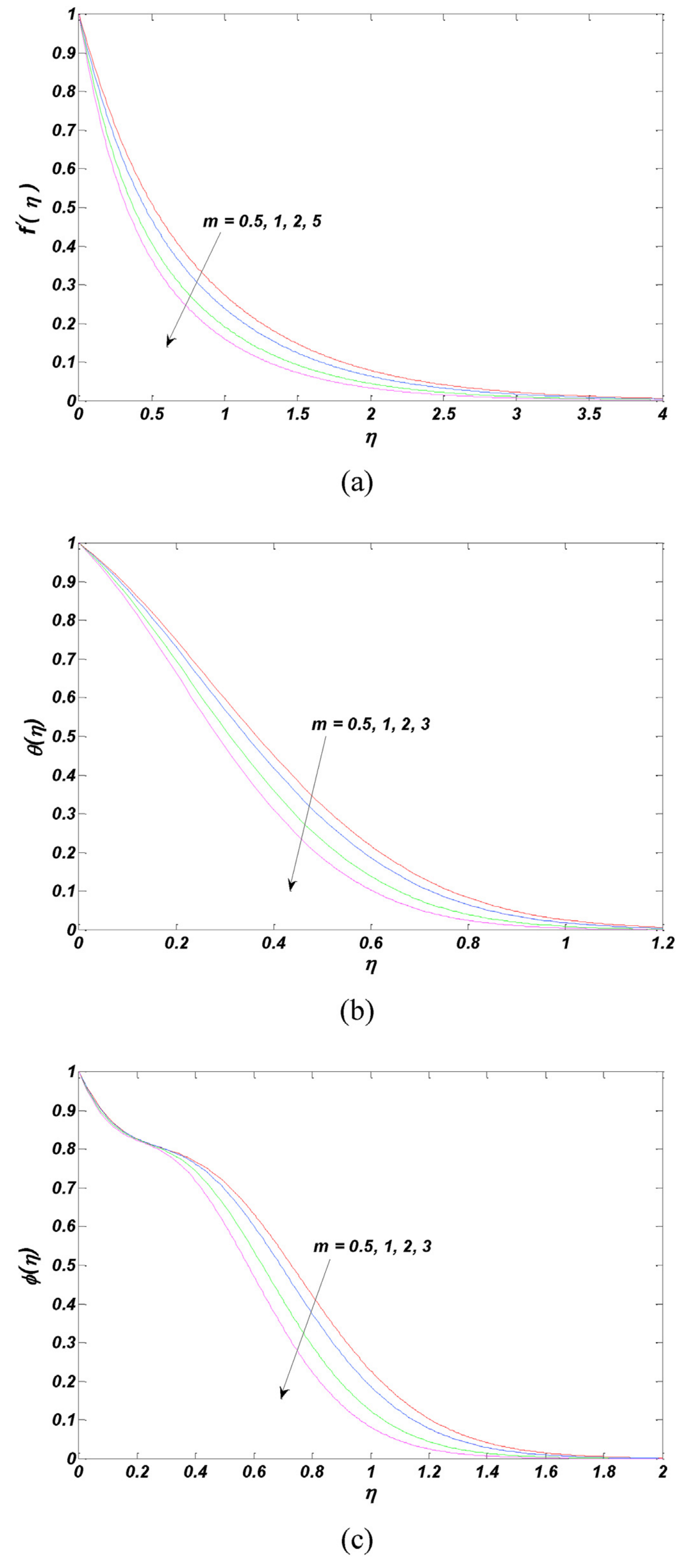


Fig. 5. Effects of m on (a) $f'(\eta)$, (b) $\theta(\eta)$ and (c) $\varphi(\eta)$.

3. Solution methodology and its validation

The system of nonlinear ODEs is numerically solved by "bvp4c" [39-42], a programme of MATLAB after converting the boundary value problem into the initial value problem, which is given by:

$$\left.\begin{aligned}
&f' = p, p' = q, \theta' = r, \varphi' = s,\\
&q' = \left[mp^2 - \left(\frac{m+3}{2}\right)fq - \left\{\frac{\theta_r}{(\theta_r-\theta)^2}\right\}qr + A\left(p+\frac{\eta}{2}q\right)\right] \Big/ \left(\frac{\theta_r}{\theta_r-\theta}\right),\\
&r' = \frac{1}{1+\varepsilon\theta}\left[-\varepsilon r^2 - \Pr\left(1-\frac{\theta}{\theta_r}\right)(1+\varepsilon\theta)\left\{\left(\frac{m+3}{2}\right)fr - \frac{\eta}{2}Ar + Nbrs + Ntr^2\right\}\right],\\
&s' = Sc\left\{\frac{\eta}{2}As - \left(\frac{m+3}{2}\right)fs\right\} - \frac{Nt}{Nb}r',
\end{aligned}\right\} \tag{21}$$

with

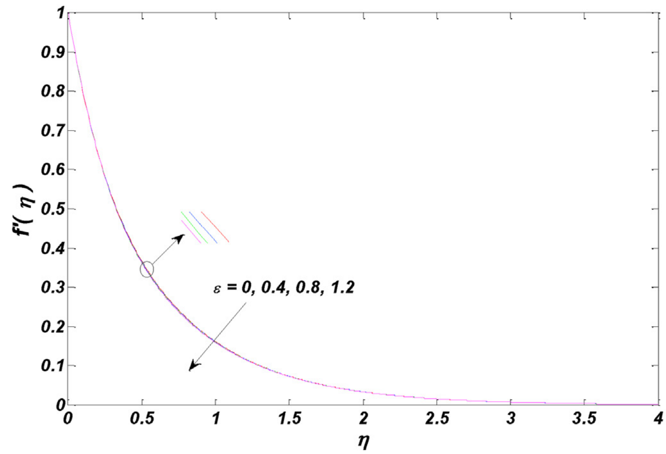
$$f(0) = 0, p(0) = 1, \theta(0) = 1, \varphi(0) = 1. \tag{22}$$

The bvp4c package is related to finite difference method which is 4^{th}-order accurate and it uses three-stage Lobatto scheme. This collocation formula provides solutions in terms of collocation polynomials which are C^1-continuous functions. Mesh points are used in this technique to divide the integration interval into subintervals. The residual error term is used to control the error of the solution and tolerance level is fixed in order 10^{-5} for convergence of the solution. A set of suitable initial approximations missing initial conditions is required to initiate the solution procedure and these guesses are selected in such a way that they fulfill the asymptotic convergence of the boundary restrictions.

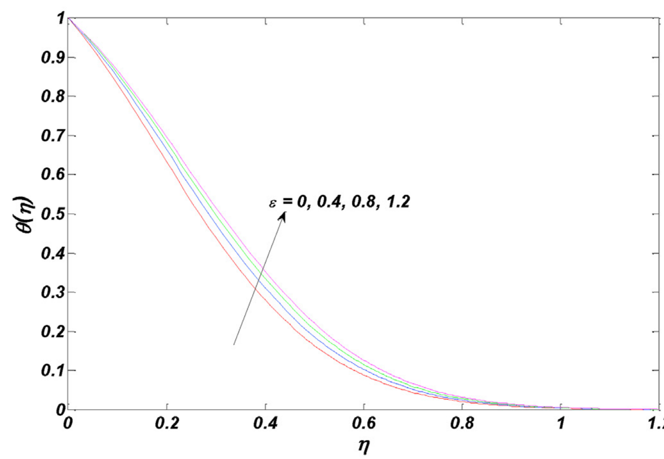
For the validation of implemented numerical scheme as well as computed results, a comparison between numerical results and analytic solution given by expression Eqs. (19) and (20) for the nonlinear expanding parameter $m = 3$ is shown in Fig. 2 for the velocity and temperature profiles. We observe that both solutions match in an excellent manner and this confirms the accuracy of the numerical method.

4. Results and discussion

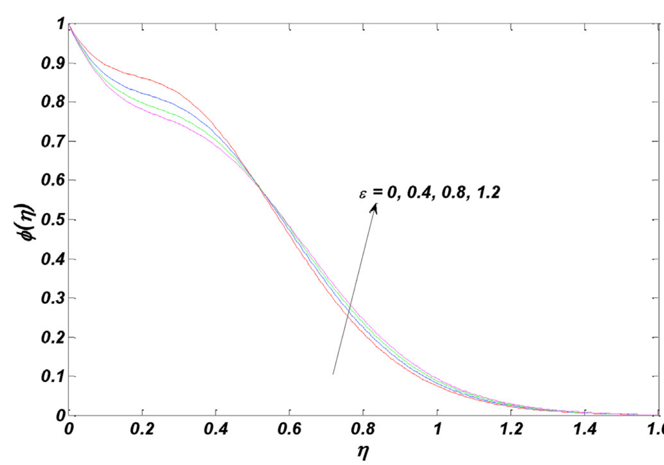
The computational outcomes explore the impact of temperature-dependent viscosity and thermal conductivity on unsteady nanofluid flow past a nonlinear radially expanding sheet. In this analysis, we consider negative values of θ_r, because θ_r is negative for liquid (here we consider water as base fluid) and positive for gases [30]. This happens, since the viscosity of liquid decays with a rise in temperature of the fluid, while the viscosity of gases enhances with temperature. For a large value of θ_r, change in viscosity due to temperature can be ignored because either γ or $T_w - T_\infty$ is very small i.e. $\theta_r \to -\infty$ reduces to constant viscosity case. But for small value of θ_r, viscosity remarkably changes with temperature. Thus, the assumption of variable viscosity is important. From the literature on temperature-dependent thermal conductivity, the domain of thermal conductivity parameter divides in different ranges based on the type of liquid which we want to consider as conducting fluid. For lubricating oil ($-1 \le \varepsilon \le 0$), water ($0 \le \varepsilon \le 1.2$) and gas ($0 \le \varepsilon \le 6$)(Das et al. [31]). Here we have taken working fluid as water, so we consider the range of ε from 0 to 1.2. A particular case $\varepsilon = 0$ is equivalent to studying fluid flow with constant thermal conductivity. The default values of the involved parameters are taken as $A = 0.1, m = 3, \theta_r = -2, Nb = 0.1, Nt = 0.2, \varepsilon = 0.4, \Pr = 7$ and $Sc = 5$ in the numerical simulation, otherwise specified. Here velocity, temperature and concentration for different values of involved parameters are analyzed through the graphical demonstrations. Also, the impact of various parameters on drag force, temperature and concentration gradients are analyzed graphically.



(a)



(b)



(c)

Fig. 6. Effects of ε on (a) $f'(\eta)$, (b) $\theta(\eta)$ and (c) $\varphi(\eta)$.

4.1. Influences on velocity, temperature and concentration

Firstly, the impacts of unsteadiness ($A = 0.1, 0.2, 0.3$) instead of the steady-state flow ($A = 0$) on nanofluid velocity, temperature and concentration (f', θ, φ) are analyzed through Fig. 3. It is noticed that velocity shows dissimilar behaviour, i.e. initially(near the surface) it decreases

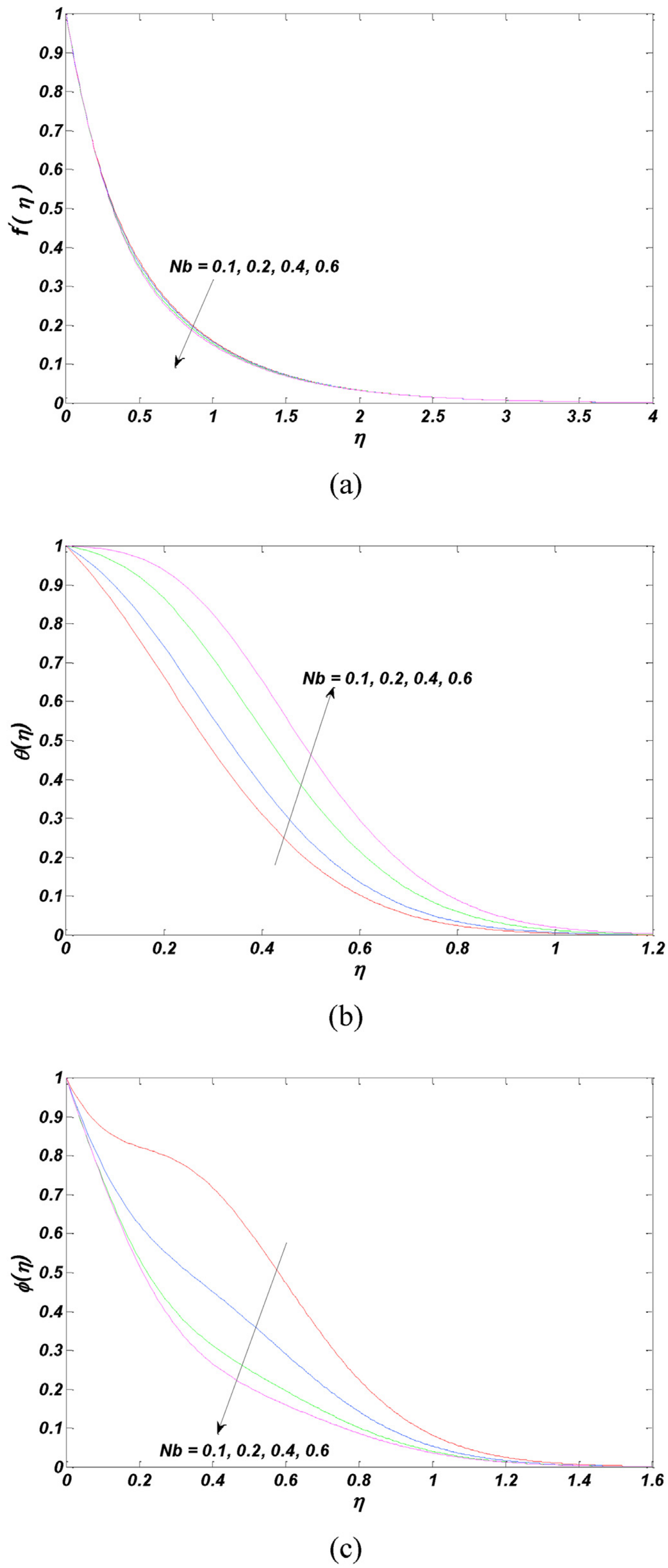


Fig. 7. Effects of *Nb* on (a) $f'(\eta)$, (b) $\theta(\eta)$ and (c) $\varphi(\eta)$.

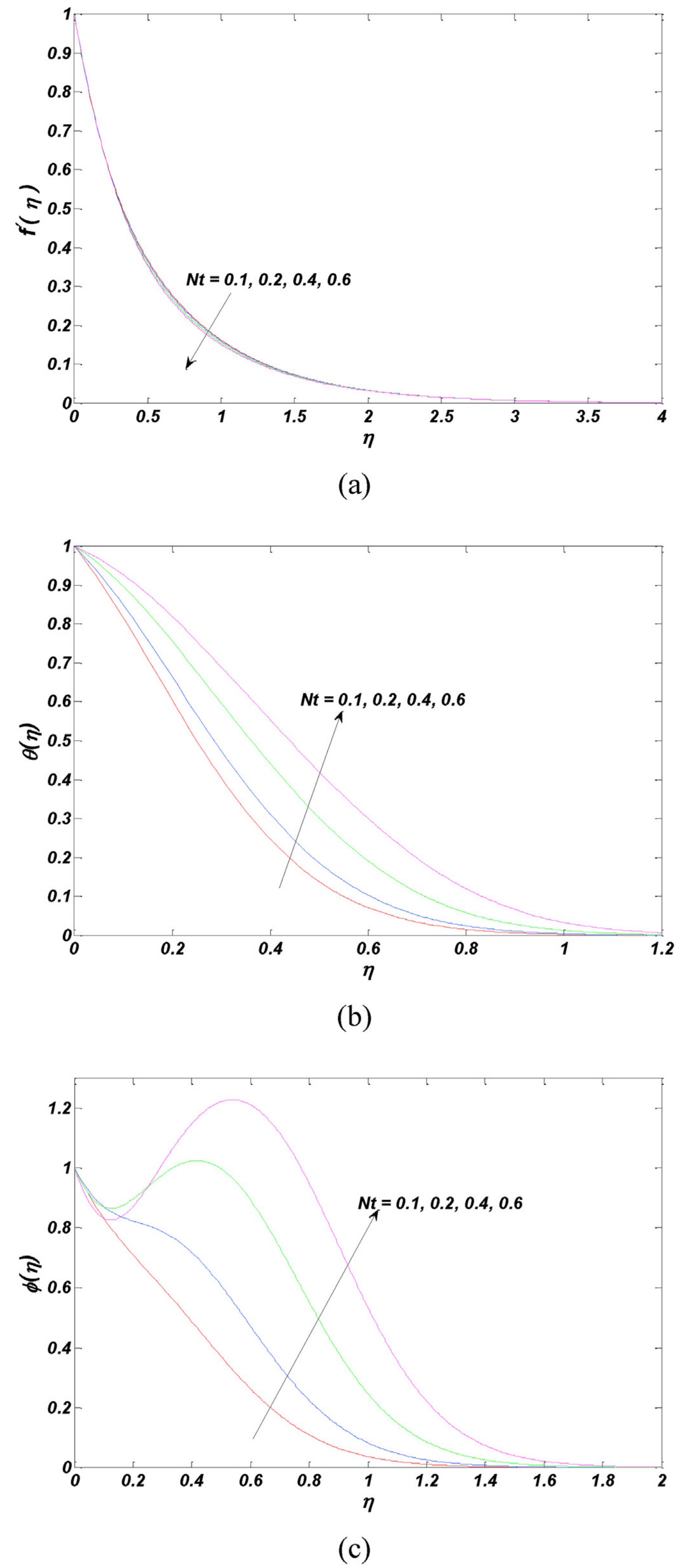


Fig. 8. Effects of *Nt* on (a) $f'(\eta)$, (b) $\theta(\eta)$ and (c) $\varphi(\eta)$.

and later(away from surface) increases with a larger estimation of unsteadiness. The escalating trends of temperature and concentration and their respective boundary layer thicknesses(BLTes) are witnessed. Alternatively, unsteadiness of the flow has capability to intensify the thickness of thermal and concentration boundary layers. Impact of flow unsteadiness is dominant near the surface.

The consequences of variable viscosity parameter ($\theta_r = -1, -2, -5, -10$) on f', θ, φ are revealed in Fig. 4. From the figure, it can be seen that in a certain range of η velocity within the boundary layer grows, while after that it decays. Physically, near the surface variation of flow velocity is more because of high temperature of fluid layers adjacent to the surface which influences viscous force. But temperature and concentration of

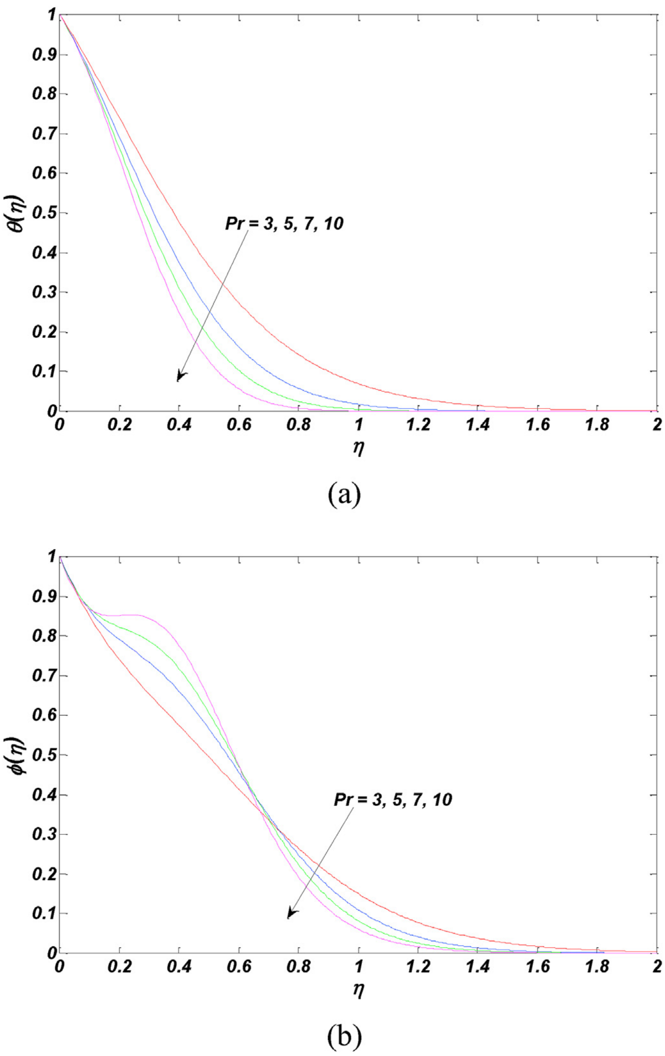


Fig. 9. Effects of Pr on (a) $\theta(\eta)$ and (b) $\varphi(\eta)$.

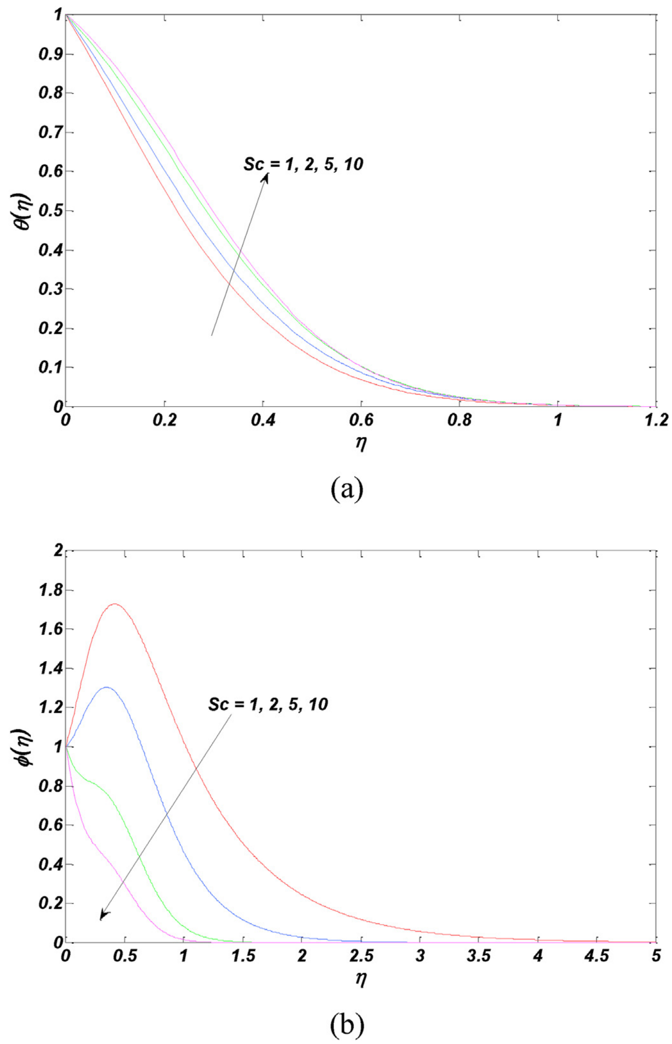


Fig. 10. Effects of Sc on (a) $\theta(\eta)$ and (b) $\varphi(\eta)$.

nanofluid and related thermal and nanoparticle concentration BLTes decay with absolute values of θ_r. It is observed from figures that the impact of the viscosity is less significant for larger values of θ_r because for higher θ_r value fluid viscosity is tending to a constant value.

Fig. 5 sketches the nature of f', θ, φ with index of expanding surface (nonlinear expanding parameter $m = 0.5, 1, 2, 3$), where $m = 1$ denotes the linear expanding surface. One can conclude from the figure that as the index of nonlinearity enhances, the nanofluid velocity, temperature and their related momentum and thermal BLTes decay. Also, a minor impact of m on the concentration of nanoparticles near the surface, but far from the surface change in nanoparticle concentration is notable, i.e. nanoparticle concentration and its BLT decreases with m. With higher values of m, the surface cooling rate and nano-mass transfer rate enhance and it works behind the declines of thermal and nanoparticle concentration BLTes.

The impacts of thermal conduction parameter ($\varepsilon = 0, 0.4, 0.8, 1.2$) on f', θ, φ is portrayed in Fig. 6. Velocity and temperature variations show opposite nature with ε, i.e. velocity decays and temperature grows with ε. Physically, thermal conductivity of the nanofluid enhances with ε which raises heat transport within the fluid, and hence improve the temperature of nanofluid. Thermal boundary layer becomes thicker for higher value of thermal conduction parameter. While concentration is treated in a quite different manner with ε. It can be noticed that in some range near the wall concentration is decreasing function of ε, while after that concentration is increasing function of ε and nanoparticle concentration boundary layer thickens with ε.

Fig. 7 is plotted to describe the impact of Brownian motion parameter ($Nb = 0.1, 0.2, 0.4, 0.6$) on f', θ, φ. Both velocity and concentration and related BLTes decay for larger estimation of Nb. Also, noted influence on concentration is prominent for lower variation of Brownian motion and its effect reduces with the more chaotic motion of nanoparticles. Nanofluid temperature and its related BLT are upsurging functions of Nb. It happens due to enhancement of higher kinetic energy of nano-sized particles for their chaotic motion.

The variation of thermophoresis parameter ($Nt = 0.1, 0.2, 0.4, 0.6$) on f', θ, φ can be seen via Fig. 8. It is observed that velocity declines for increasing value of Nt while nanofluid temperature, nanoparticle concentration and their BLTes are uplifting functions of estimated parameter Nt. Also, concentration overshoot is observed for higher Nt. Thermophoresis force is a temperature gradient force that causes transportation of hotter molecules having more kinetic energy towards the low temperature region. Therefore, the nanoparticle concentration and fluid temperature increase with the thermophoretic force. Applications of such force can be seen in oil industry, thermal precipitators, polymeric molecules transportation in micro-sized channels, in controlling DNA transportation and in many other medical phenomena [43].

The effect of Prandtl number ($\mathrm{Pr} = 3, 5, 7, 10$) on temperature and concentration is examined via Fig. 9. Enlarging values of Pr relates to lower temperature and thermal BLT. This happens due to slow thermal diffusion for the higher Pr value. Larger estimations of Pr indicating slightly different changes on the concentration of nanoparticles, i.e. near

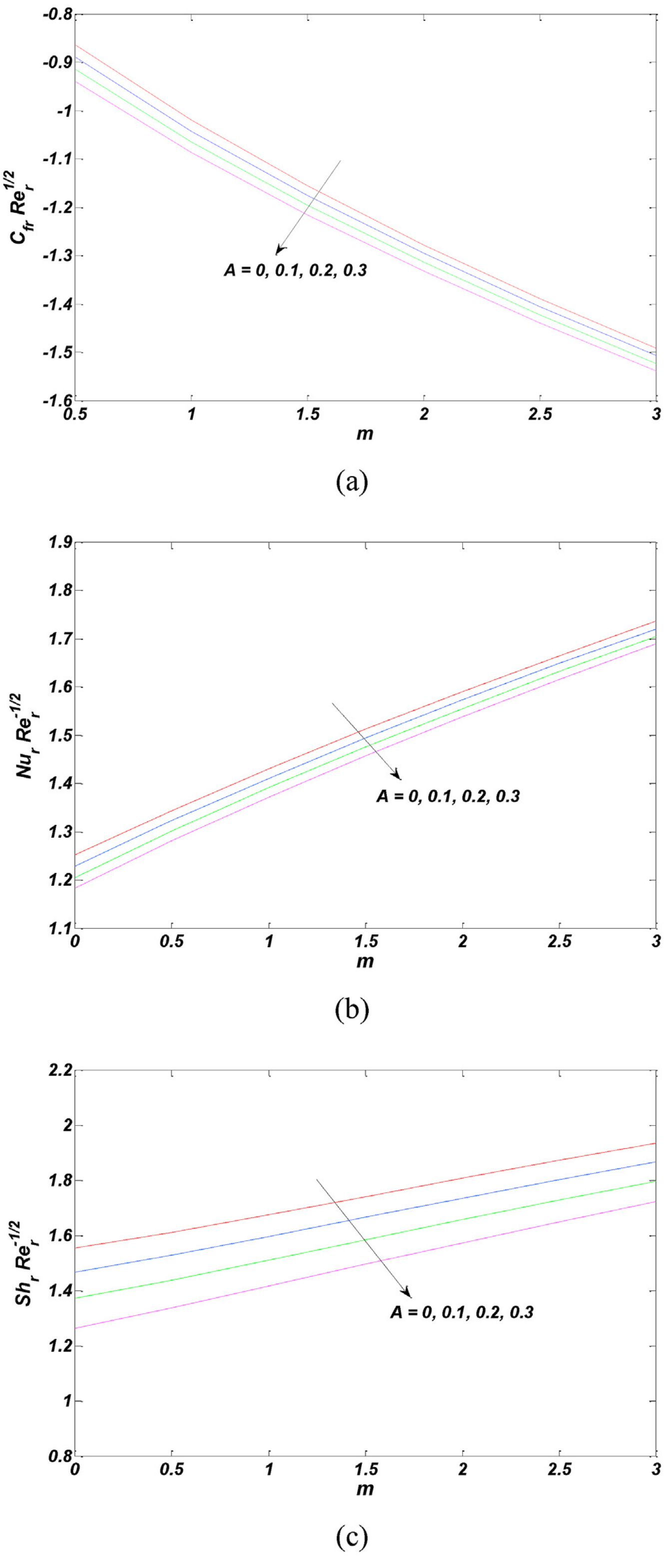


Fig. 11. Effects of A and m on (a) skin-friction coefficient, (b) Nusselt number and (c) Sherwood number.

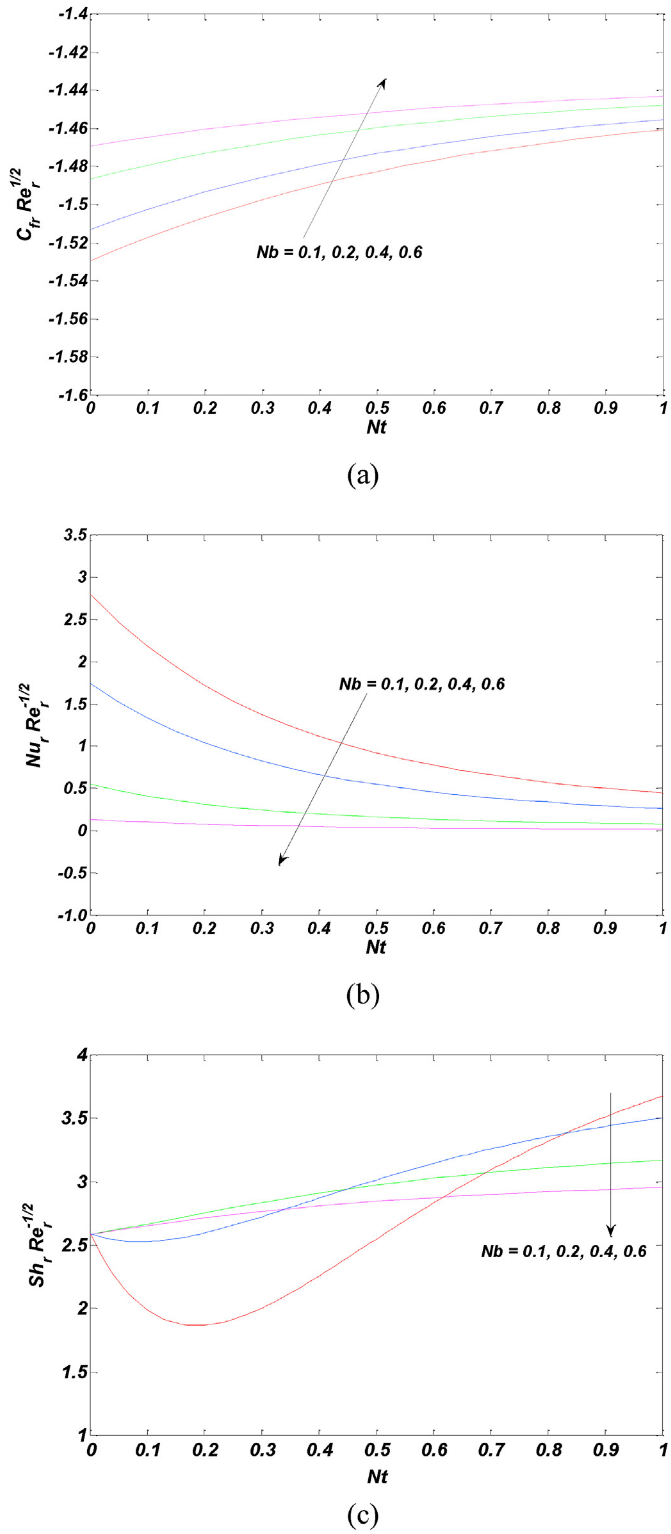


Fig. 12. Effects of Nb and Nt on (a) skin-friction coefficient, (b) Nusselt number and (c) Sherwood number.

the surface concentration is increasing function of Pr, but far from the surface, it is decreasing function of Pr. Also, note that the concentration BLT becomes thinner with the large estimation of Pr.

The variation of θ, φ for the estimation of Schmidt number ($Sc = 1, 2, 5, 10$) can be measured from Fig. 10. We can note that the temperature and thermal BLT is enlarging function of Sc. The figure also demonstrates the effect of Sc on the concentration of nanoparticles. From this figure, concentration overshoot is noted for smaller values of Sc. Concentration and its associated BLT reduce with Sc. Schmidt number is measured as the relative thickness of the momentum and concentration boundary layers. When Sc is small, mass diffusion is more than the momentum. So, the concentration boundary layer is thicker as compared to the momentum boundary layer.

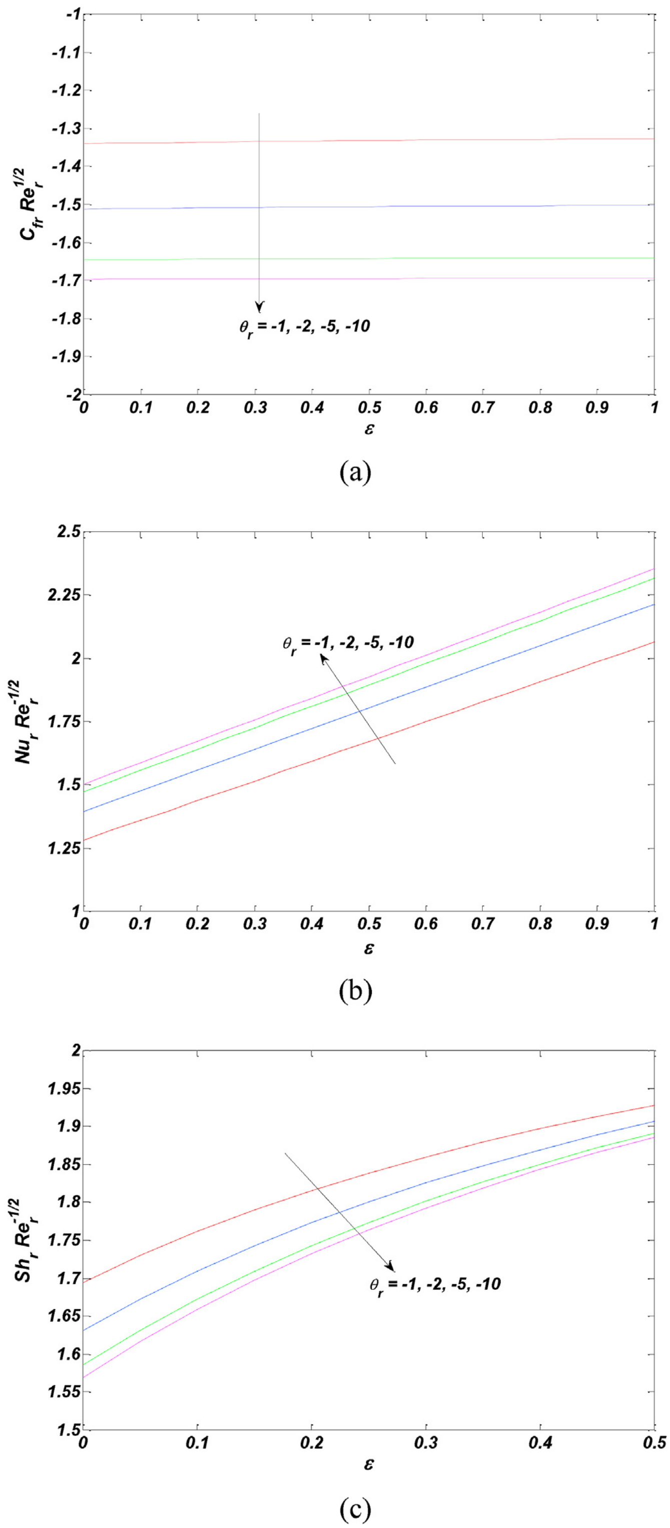


Fig. 13. Effects of θ_r and ε on (a) skin-friction coefficient, (b) Nusselt number and (c) Sherwood number.

4.2. Influences on skin-friction coefficient, Nusselt and Sherwood numbers(C_{fr}, Nu_r, Sh_r)

In this section, we described that how physical quantities, like skin-friction, Nusselt number and Sherwood number respond with the variations of unsteady parameter, nonlinearity expanding index, temperature-dependent viscosity and conductivity parameter, Brownian and thermophoresis parameter.

The influence of unsteadiness and nonlinearity of expansion of the surface on the surface drag, heat and nano-mass transfer rate can be understood from Fig. 11. We can conclude that unsteadiness of the flow reduces the surface drag, heat transfer rate and nano-mass transfer rate. While the rates of heat and nano-mass transfer increase with the nonlinearity expanding index, but surface drag reduces.

From Fig. 12, one can observe that both *Nb* and *Nt* show the same effect for drag force, heat transfer and nano-mass transfer rates. Surface drag is an increasing function of *Nb* as well as *Nt*. Enhancement in random movement of nanoparticles due to Brownian motion and migration of small particles from hot to cold region caused by temperature gradient force create friction and hence drag force increases. Whereas, the rate of heat transfer decreases with both *Nb* and *Nt*. Also, it is noted that for lower values of Brownian motion parameter, heat transfer rate changes significantly with *Nt*. From the figure, we can also state that nano-mass transfer rate decreases with *Nb* for strong thermophoresis force. Also, the nano-mass transfer rate reduces with *Nb* for smaller values of *Nt*. So, for minor Brownian motion the nano-mass transfer rate diminishes with increasing thermophoresis force having small magnitude. But, for stronger thermophoresis force, nano-mass transfer rate grows.

The effect of ε and θ_r on the surface drag, heat and nano-mass transfer rates are discussed through Fig. 13. From the figure, a minute change can be noted in the surface drag for higher values of the thermal conductivity parameter. But surface drag decreases prominently with the increasing absolute value of θ_r. The figure also reveals that the rate of heat transfer enhances with both ε and θ_r. Whereas, for higher values of thermal conductivity parameter, the rate of nano-mass transfer is high, and opposite effect is observed for θ_r, i.e. the nano-mass transfer rate reduces for small value of θ_r. In engineering processes, where heat is generated due to internal friction, it is necessary to take into account the effect of temperature on viscosity and thermal conductivity.

5. Main outcomes

Unsteady nanofluid boundary layer flow generated by a radially nonlinear expanding surface is mathematically represented by the Buongiorno model. In this study, viscosity and thermal conductivity of the fluid are considered as a function of temperature. Numerical computation is performed to solve the problem, and to demonstrate the effect of influential parameters graphically. The study reveals that flow unsteadiness produces growths of nanofluid temperature and nanoparticle concentration inside boundary layer, whereas these two exhibits reducing trend with rise of variable viscosity related parameter. In addition, the momentum, thermal and concentration boundary layers become thinner with nonlinear expanding parameter related to expansion of the surface. While the thermal BLT is small for a larger value of variable thermal conductivity related parameter. Also, for less prominent Brownian motion, the nano-mass transfer rate reduces with increasing thermophoresis force with small magnitude and for stronger thermophoresis force, it grows. Furthermore, the rates of heat and nano-mass transfers enhance with variable thermal conductivity parameter. It is noticed that the surface drag decreases with both unsteadiness of the flow and nonlinear expanding index, whereas the heat and nano-mass transfer rates enhance with nonlinear expanding index and decay with flow unsteadiness. Finally, the nanofluid heat transfer rate increases with variable viscosity related parameter, while surface drag and nano-mass transfer rate decrease.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The work of S. Rajput [09/013(0843)/2018-EMR-I] is supported by CSIR, India and K. Bhattacharyya acknowledges "Faculty Incentive Grant" (No. R/Dev/D/IoE/Incentive/2022-23/47686) of Banaras Hindu University. The authors are thankful to the esteemed reviewers for their very useful suggestions.

References

[1] Choi, SUS. Enhancing thermal conductivity of fluids with nanoparticle. In: Siginer DA, Wang HP, editors. Developments and Applications of Non-newtonian Flows, ASME. vol. 66, pp. 99-105.
[2] R. Saidur, K.Y. Leong, H.A. Mohammed, A review on applications and challenges of nanofluids, Renew. Sustain. Energy Rev. 15 (3) (2011) 1646–1668.
[3] R.K. Tiwari, M.K. Das, Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids, Int. J. Heat Mass Tran. 50 (9-10) (2007) 2002–2018.
[4] J. Buongiorno, Convective transport in nanofluids, J. Heat Tran. 50 (2006) 240–250.
[5] W.A. Khan, I. Pop, Boundary-layer flow of a nanofluid past a stretching sheet, Int. J. Heat Mass Tran. 53 (11-12) (2010) 2477–2483.
[6] A.V. Kuznetsov, D.A. Nield, Natural convective boundary-layer flow of a nanofluid past a vertical plate: a revised model, Int. J. Therm. Sci. 77 (2014) 126–129.
[7] K. Bhattacharyya, G.C. Layek, Magnetohydrodynamic boundary layer flow of nanofluid over an exponentially stretching permeable sheet, Phys. Res. Int. (2014), 592536, 2014.
[8] A. Hussanan, M.Z. Salleh, I. Khan, Z.M. Chen, Unsteady water functionalized oxide and non-oxide nanofluids flow over an infinite accelerated plate, Chin. J. Phys. 62 (2019) 115–131.
[9] M. Ramzan, T. Mehmood, H. Alotaibi, H.A.S. Ghazwani, T. Muhammad, Comparative study of hybrid and nanofluid flows amidst two rotating disks with thermal stratification: statistical and numerical approaches, Case Stud. Therm. Eng. 28 (2021), 101596.
[10] A.K. Pandey, S. Rajput, K. Bhattacharyya, P. Sibanda, Impact of metal oxide nanoparticles on unsteady stagnation point flow of the hybrid base fluid along a flat surface, Pramana 95 (1) (2021) 1–9.
[11] M. Hussain, U. Farooq, M. Sheremet, Nonsimilar convective thermal transport analysis of EMHD stagnation Casson nanofluid flow subjected to particle shape factor and thermal radiations, Int. Commun. Heat Mass Tran. 137 (2022), 106230.
[12] A.K. Verma, S. Rajput, K. Bhattacharyya, A.J. Chamkha, Nanoparticle's radius effect on unsteady mixed convective copper-water nanofluid flow over an expanding sheet in porous medium with boundary slip, Chem. Eng. J. Adv. 12 (2022), 100366.
[13] B.C. Sakiadis, Boundary-layer behavior on continuous solid surfaces: I. Boundary-layer equations for two-dimensional and axisymmetric flow, AIChE J. 7 (1) (1961) 26–28.
[14] P.D. Ariel, Axisymmetric flow due to a stretching sheet with partial slip, Comput. Math. Appl. 54 (7-8) (2007) 1169–1183.
[15] M. Sajid, T. Hayat, S. Asghar, K. Vajravelu, Analytic solution for axisymmetric flow over a nonlinearly stretching sheet, Arch. Appl. Mech. 78 (2) (2008) 127–134.
[16] A. Shahzad, R. Ali, M. Khan, On the exact solution for axisymmetric flow and heat transfer over a nonlinear radially stretching sheet, Chin. Phys. Lett. 29 (8) (2012), 084705.
[17] U. Khan, A. Zaib, A. Ishak, N.C. Roy, S.A. Bakar, T. Muhammad, A.H. Abdel-Aty, I.S. Yahia, Exact solutions for MHD axisymmetric hybrid nanofluid flow and heat transfer over a permeable non-linear radially shrinking/stretching surface with mutual impacts of thermal radiation, Eur. Phys. J. Spec. Top. 231 (2022) 1195–1204.
[18] M. Mustafa, J.A. Khan, T. Hayat, A. Alsaedi, Analytical and numerical solutions for axisymmetric flow of nanofluid due to non-linearly stretching sheet, Int. J. Non Lin. Mech. 71 (2015) 22–29.
[19] A. Shahzad, J. Ahmed, M. Khan, On heat transfer analysis of axisymmetric flow of viscous fluid over a nonlinear radially stretching sheet, Alex. Eng. J. 55 (3) (2016) 2423–2429.
[20] A. Shahzad, R. Ali, M. Hussain, M. Kamran, Unsteady axisymmetric flow and heat transfer over time-dependent radially stretching sheet, Alex. Eng. J. 56 (1) (2017) 35–41.
[21] M. Azam, M. Khan, A.S. Alshomrani, Effects of magnetic field and partial slip on unsteady axisymmetric flow of Carreau nanofluid over a radially stretching surface, Results Phys. 7 (2017) 2671–2682.
[22] M. Azam, F. Mabood, T. Xu, M. Waly, I. Tlili, Entropy optimized radiative heat transportation in axisymmetric flow of Williamson nanofluid with activation energy, Results Phys. 19 (2020), 103576.
[23] Y.Q. Song, F. Shah, S.A. Khan, M.I. Khan, M.Y. Malik, T.C. Sun, Irreversibility analysis for axisymmetric nanomaterial flow towards a stretched surface, Chaos, Solit. Fractals 150 (2021), 111145.
[24] R. Ellahi, M. Raza, K. Vafai, Series solutions of non-Newtonian nanofluids with Reynolds' model and Vogel's model by means of the homotopy analysis method, Math. Comput. Model. 55 (7-8) (2012) 1876–1891.
[25] J.X. Ling, A. Dybbs, Forced Convection over a Flat Plate Submersed in a Porous Medium: Variable Viscosity Case (No. CONF-871234-), American Society of Mechanical Engineers, New York, NY, 1987.
[26] D. Knežević, V. Savić, Mathematical modeling of changing of dynamic viscosity, as a function of temperature and pressure, of mineral oils for hydraulic systems, Facta Univ. – Ser. Mech. Eng. 4 (2006) 27–34.
[27] K. Das, N. Acharya, P.K. Kundu, Influence of Variable Fluid Properties on Nanofluid Flow over a Wedge with Surface Slip, in: Arabian Journal for Science & Engineering, vol. 43, Springer Science & Business Media BV, 2018, pp. 2119–2131, 5.
[28] W.M. Kays, Convective Heat and Mass Transfer, Tata McGraw-Hill Education, 1966, p. 362.
[29] T.C. Chiam, Heat transfer in a fluid with variable thermal conductivity over a linearly stretching sheet, Acta Mech. 129 (1) (1998) 63–72.
[30] F.C. Lai, F.A. Kulacki, The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium, Int. J. Heat Mass Tran. 33 (5) (1990) 1028–1031.
[31] K. Das, M.T. Sk, P.K. Kundu, Steady nanofluid flow with variable fluid possessions over a linearly extending surface: a Lie group exploration, Alex. Eng. J. 57 (1) (2018) 415–425.
[32] B. Ali, R.A. Naqvi, Y. Nie, S.A. Khan, M.T. Sadiq, A.U. Rehman, S. Abdal, Variable viscosity effects on unsteady mhd an axisymmetric nanofluid flow over a stretching surface with thermo-diffusion: fem approach, Symmetry 12 (2) (2020) 234.
[33] T.G. Myers, J.P.F. Charpin, M.S. Tshehla, The flow of a variable viscosity fluid between parallel plates with shear heating, Appl. Math. Model. 30 (9) (2006) 799–815.
[34] K.V. Prasad, K. Vajravelu, P.S. Datti, The effects of variable fluid properties on the hydro-magnetic flow and heat transfer over a non-linearly stretching sheet, Int. J. Therm. Sci. 49 (3) (2010) 603–610.
[35] K. Das, Influence of thermophoresis and chemical reaction on MHD micropolar fluid flow with variable fluid properties, Int. J. Heat Mass Tran. 55 (23-24) (2012) 7166–7174.
[36] M.I. Khan, S.U. Khan, M. Jameel, Y.M. Chu, I. Tlili, S. Kadry, Significance of temperature-dependent viscosity and thermal conductivity of Walter's B nanoliquid when sinusodal wall and motile microorganisms density are significant, Surface. Interfac. 22 (2021), 100849.
[37] N.S. Akbar, E.N. Maraj, N.F.M. Noor, M.B. Habib, Exact solutions of an unsteady thermal conductive pressure driven peristaltic transport with temperature-dependent nanofluid viscosity, Case Stud. Therm. Eng. 35 (2022), 102124.
[38] M.O. Lawal, K.B. Kasali, H.A. Ogunseye, M.O. Oni, Y.O. Tijani, Y.T. Lawal, On the mathematical model of Eyring–Powell nanofluid flow with non-linear radiation, variable thermal conductivity and viscosity, Partial Differential Equat. Appl. Mathematics 5 (2022), 100318.
[39] M. Khan, M. Irfan, L. Ahmad, W.A. Khan, Simultaneous investigation of MHD and convective phenomena on time-dependent flow of Carreau nanofluid with variable properties: dual solutions, Phys. Lett. 382 (34) (2018) 2334–2342.
[40] T. Sajid, W. Jamshed, F. Shahzad, E.K. Akgül, K.S. Nisar, M.R. Eid, Impact of gold nanoparticles along with Maxwell velocity and Smoluchowski temperature slip boundary conditions on fluid flow: sutterby model, Chin. J. Phys. 77 (2022) 1387–1404.
[41] S. Rajput, A.K. Verma, K. Bhattacharyya, A.J. Chamkha, Unsteady nonlinear mixed convective flow of nanofluid over a wedge: Buongiorno model, Waves Random Complex Media (2021), https://doi.org/10.1080/17455030.2021.1987586.
[42] T. Salahuddin, N. Siddique, M. Khan, M. Altanji, A significant study on flow analysis of viscoelastic fluid with variable thermo-physical properties, Math. Comput. Simulat. 194 (2021) 416–429.
[43] S. Ramachandran, C.B. Sobhan, G.P. Peterson, Thermophoresis of nanoparticles in liquids, Int. J. Heat Mass Tran. 147 (2020), 118925.